J. math. fluid mech. 9 (2007) 295–342
1422-6928/07/030295-48

DOI 10.1007/s00021-005-0200-8

Journal of Mathematical Fluid Mechanics

Downstream Asymptotics in Exterior Domains: from Stationary Wakes to Time Periodic Flows

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Communicated by G. P. Galdi

Abstract. In this paper, we consider the time-dependent Navier–Stokes equations in the half-space $[x_0,\infty)\times\mathbf{R}\subset\mathbf{R}^2$, with boundary data on the line $x=x_0$ assumed to be time-periodic (or stationary) with a fixed asymptotic velocity $\mathbf{u}_\infty=(1,0)$ at infinity. We show that there exist (locally) unique solutions for all data satisfying a center-stable manifold compatibility condition in a certain class of functions. Furthermore, we prove that as $x\to\infty$, the vorticity decomposes itself in a dominant stationary part on the parabolic scale $y\sim\sqrt{x}$ and corrections of order $x^{-\frac{3}{2}+\varepsilon}$, while the velocity field decomposes itself in a dominant stationary part in form of an explicit multi-scale expansion on the scales $y\sim\sqrt{x}$ and $y\sim x$ and corrections decaying at least like $x^{-\frac{9}{8}+\varepsilon}$. The asymptotic fields are made of linear combinations of universal functions with coefficients depending mildly on the boundary data. The asymptotic expansion for the component parallel to $\mathbf{u}_\infty$ contains 'non-trivial' terms in the parabolic scale with amplitude $\ln(x)x^{-1}$ and x^{-1}. To first order, our results also imply that time-periodic wakes behave like stationary ones as $x\to\infty$.

The class of functions used to prove these results is 'natural' in the sense that the well known 'Physically Reasonable' (in the sense of Finn & Smith) stationary solutions of the Navier–Stokes equations around an obstacle fall into that class if the half-space extends in the downstream direction and the boundary ($x=x_0$) is sufficiently far downstream. In that case, the coefficients appearing in the asymptotics can be linearly related to the net force acting on the obstacle. In particular, the asymptotic description holds for 'Physically Reasonable' stationary solutions in exterior domains, *without restrictions on the size of the drag acting on the obstacle*. To our knowledge, it is the first time that estimates uncovering the $\ln(x)x^{-1}$ correction are proved in this setting.

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Supported by the Fonds National Suisse.

1. Introduction

1.1. Informal presentation of the results

In this paper, we consider the time-dependent Navier–Stokes equations

$$\begin{aligned} &\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = \frac{1}{\mathrm{Re}} \triangle \mathbf{u} - \nabla p, && \nabla \cdot \mathbf{u} = 0, \\ &\mathbf{u}(x,y,t)|_{x=x_0} = \mathbf{u}_b(y,t), && \lim_{x^2+y^2 \to \infty} \mathbf{u}(x,y,t) = \mathbf{u}_\infty \equiv \begin{pmatrix} u_\infty \\ 0 \end{pmatrix} \end{aligned} \tag{1.1}$$

in the half-space $\Omega_+ = [x_0, \infty) \times \mathbf{R}$, with time-periodic boundary data $\mathbf{u}_b(y,t) = \sum_{n \in \mathbf{Z}} e^{in\tau t} \mathbf{u}_{b,n}(y)$ where $\tau > 0$ is the basic frequency (Strouhal number) and $\mathbf{u}_b \in l^1(\mathbf{Z}, \mathcal{B})$ for some functional space $\mathcal{B}$ to be defined later on. With appropriate scalings, we can set $|\mathbf{u}_\infty| = u_\infty = 1$ and $\mathrm{Re} = 1$ without loss of generality. The scale of the Reynolds number then translates to the scale of $\mathbf{u}_b$.

We interpret this problem as a simplified version of the 'usual' exterior problem around an obstacle

$$\begin{aligned} &\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = \frac{1}{\mathrm{Re}} \triangle \mathbf{u} - \nabla p, && \nabla \cdot \mathbf{u} = 0, \\ &\mathbf{u}(x,y,t)|_{\partial\Omega} = 0, && \lim_{x^2+y^2 \to \infty} \mathbf{u}(x,y,t) = \mathbf{u}_\infty \equiv \begin{pmatrix} u_\infty \\ 0 \end{pmatrix} \end{aligned} \tag{1.2}$$

in $\mathbf{R}^2 \backslash \Omega$, where $\partial\Omega$, the obstacle, is compact and connected. Getting rid of the obstacle by considering the flow only in the downstream region is a brutal simplification. We hope that by capturing the main difficulty of (1.2), (the spatial asymmetry introduced by $\mathbf{u}_\infty$, as seen in the slow decay of vorticity as $x \to \infty$ for instance), techniques used in this paper could shed a new light on the theory on the Navier–Stokes equations (1.2) which began with J. Leray's pioneering work in the 1930's (see also [7] and references therein).

The question we address here is to give a quantitative description of the flow in the so-called 'wake region' which extends downwards of the obstacle (i.e. as $x \to \infty$). In previous papers [9, 18] such a description has been obtained in the stationary case by assuming that the restriction of the solution of (1.2) on a given line $x = x_0 \gg 1$ was in a certain function class. Unfortunately, the function class used in these papers is rather unorthodox and the question whether the (restriction of) solutions of (1.2) were in this class was completely left open.

Leaving this question aside for the moment, it follows from [9, 18] that as $x \to \infty$, the velocity field $\mathbf{u}$ and the vorticity $\omega = \nabla \times \mathbf{u}$ satisfy

$$\begin{aligned} \mathbf{u}(x,y) &= \mathbf{u}_\infty + \left(\begin{smallmatrix} \tilde{u}_{\mathbf{a}}(x,y) \\ \tilde{v}_{\mathbf{a}}(x,y) \end{smallmatrix}\right) + \mathcal{O}(x^{-1+\varphi_0}), \\ \omega(x,y) &= \omega_{\mathbf{a}}(x,y) + \mathcal{O}(x^{-\frac{3}{2}+\varphi_0}), \end{aligned} \tag{1.3}$$

for some $\varphi_0 > 0$ and parameters $\mathbf{a} = (a_1, a_2, a_3)$, where,

$$\tilde{u}_{\mathbf{a}}(x,y) = \tfrac{a_1}{\sqrt{x}} f_0(\tfrac{y}{\sqrt{x}}) + \tfrac{1}{x}\Big(a_2 g_0(\tfrac{y}{x}) - a_3 g_1(\tfrac{y}{x})\Big) \tag{1.4}$$

$$\tilde{v}_{\mathbf{a}}(x,y) = \tfrac{a_1}{2x} f_1(\tfrac{y}{\sqrt{x}}) + \tfrac{1}{x}\Big(a_2 g_1(\tfrac{y}{x}) + a_3 g_0(\tfrac{y}{x})\Big) \tag{1.5}$$

$$\omega_{\mathbf{a}}(x,y) = \tfrac{a_1}{2x} f_1(\tfrac{y}{\sqrt{x}}), \tag{1.6}$$

with $f_m(z) = \frac{z^m e^{-\frac{z^2}{4}}}{\sqrt{4\pi}}$ and $g_m(z) = \frac{1}{\pi}\frac{z^m}{1+z^2}$.

This result was expected to hold for a long time, see e.g. [2]. It should be noted that the terms on the $y \sim x$ scale are of smaller order than the stated correction order. It is however argued in [2, 9, 18] that on the given scales ($y \sim x$ or $y \sim \sqrt{x}$) the velocity field indeed converges to its asymptotic form and furthermore that the upstream asymptotics ($x \to -\infty$ is given by (1.4) and (1.5) with $a_1 = 0$ *and the same coefficients* a_2 and a_3 as in the downstream direction. Integration of the equations (1.2) in the domain comprised between the lines $x = -x_0 \ll 0$ and $x = x_0 \gg 0$ then implies (see e.g. Appendix II in [21]) in the limit $x_0 \to \infty$ that $a_1 + 2a_2 = 0$ (mass conservation), and that the force $\mathbf{F}$ acting on the obstacle is given by

$$\mathbf{F} = \begin{pmatrix} 2a_2 \\ -2a_3 \end{pmatrix} + \int_{\mathbf{R}^2\setminus\Omega} \mathrm{d}x\mathrm{d}y\ \partial_t \mathbf{u}(x,y,t) \equiv \begin{pmatrix} \text{drag} \\ \text{lift} \end{pmatrix}, \tag{1.7}$$

which shows that for stationary flows, the parameters a_1, a_2 and a_3 are linearly related to the drag and lift acting on the obstacle. In particular, the first order asymptotics in the wake are completely determined by the net force acting on the obstacle (see also [2, 9, 18] for more physical interpretations).

For completeness, we note that (1.4) and (1.5) can be easily derived heuristically in the two following steps. First, the field $(\tilde{u}_{\mathbf{a}}, \tilde{v}_{\mathbf{a}})$ with $a_1 = 0$ would be a solution of the Navier–Stokes equations (1.1) or (1.2) (for an appropriate pressure) but for the boundary conditions. And then, as we may expect that $\partial_x^2\omega \ll \partial_x\omega$ as $x \to \infty$, the vorticity satisfies (to first order) $\partial_x\omega = \partial_y^2\omega$, whose solutions corresponding to decaying velocity fields behave asymptotically like (1.6). It is then easy to see using $\omega \approx -\partial_y u$ and $\partial_y v = -\partial_x u$ that the corresponding velocity fields are as stated in (1.4) and (1.5).

As we noted above, the results of [9, 18] suffer from two weaknesses: it is not known whether they apply to solutions of (1.2), and some terms in the asymptotic description are of smaller order than the (uncontrolled) error terms. Moreover, it is well known from experiences and numerical simulations that stationary solutions of (1.2) in exterior domains are only stable at low Reynolds numbers, and it is commonly believed (see e.g. [5, 14, 16, 17]) that at a (first) critical Reynolds number, the stationary flow loses its stability through a Hopf bifurcation and becomes time-periodic before eventually leading for larger Reynolds number to von Karman's vortex street and then to turbulence.

In this paper, we will give a more detailed asymptotic description than (1.3), as we will prove that in both the stationary and time-periodic case, the solutions of (1.1) satisfy

$$\begin{aligned}\mathbf{u}(x,y,t) &= \mathbf{u}_\infty + \begin{pmatrix} u_{\mathbf{a}(t)}(x,y) \\ v_{\mathbf{a}(t)}(x,y)\end{pmatrix} + \mathcal{O}\begin{pmatrix} x^{-\frac{9}{8}+\varphi_0} \\ x^{-\frac{3}{2}+\varphi_0}\end{pmatrix}, \\ \omega(x,y) &= \omega_{\mathbf{a}(t)}(x,y) + \mathcal{O}(x^{-\frac{3}{2}+\varphi_0}),\end{aligned} \tag{1.8}$$

uniformly in time, where $0 < \varphi_0 < \frac{1}{8}$, $\mathbf{a}(t) = (a_1, a_2(t), a_3(t), a_4, a_5, a_6)$ for some constants a_1, a_4, a_5 and a_6 and *time periodic functions* a_2 and a_3, $\omega_{\mathbf{a}}$ is as above and

$$\begin{aligned}u_{\mathbf{a}(t)}(x,y) &= \tfrac{a_1}{\sqrt{x}} f_0(\tfrac{y}{\sqrt{x}}) + \tfrac{1}{x}\Big(a_2(t) g_0(\tfrac{y}{x}) - a_3(t) g_1(\tfrac{y}{x})\Big) \\ &\quad - \tfrac{1}{2x}\Big((a_4 + a_6 \ln(x)) f_1(\tfrac{y}{\sqrt{x}}) + a_5 h(\tfrac{y}{\sqrt{x}})\Big) \\ v_{\mathbf{a}(t)}(x,y) &= \tfrac{a_1}{2x} f_1(\tfrac{y}{\sqrt{x}}) + \tfrac{1}{x}\Big(a_2(t) g_1(\tfrac{y}{x}) + a_3(t) g_0(\tfrac{y}{x})\Big)\end{aligned} \tag{1.9}$$

where

$$f_m(z) = \tfrac{z^m e^{-\frac{z^2}{4}}}{\sqrt{4\pi}}, \qquad g_m(z) = \tfrac{1}{\pi}\tfrac{z^m}{1+z^2}, \qquad h(z) = f_0(z)^2 + \tfrac{1}{8\sqrt{\pi}}\, z \operatorname{erf}\big(\tfrac{z}{2}\big) e^{-\frac{z^2}{4}}.$$

By the use of functional spaces more adapted than in [9, 18], we will prove that existing results (see e.g. [7]) on stationary solutions of (1.2) imply that (1.8) also holds for such solutions. Though we believe it should also hold for time periodic solutions of (1.2) just after the Hopf bifurcation, this question is left open in this present work, as we are not aware of any rigorous treatment of the exterior time periodic problem in 2D (see however [15] for some rigorous work on the 3D case).

In analogy with the stationary case, we may also expect that for the solution of (1.2), the asymptotic velocity field upstream ($x \to -\infty$) is given by (1.9) with $a_1 = a_4 = a_5 = a_6 = 0$ and the same coefficients $a_2(t)$ and $a_3(t)$ than in the downstream direction. If this holds, then integrating $\nabla \cdot \mathbf{u} = 0$ and $\omega = \nabla \times \mathbf{u}$ in the domain comprised between $x = -x_0 \ll 0$ and $x = x_0 \gg 0$, we get (letting $x_0 \to \infty$) $a_2(t) = -\frac{1}{2}a_1$ and $a_3(t) = \int_{\mathbf{R}^2\setminus\Omega} \omega(x,y,t)\,\mathrm{d}x\mathrm{d}y$. As this last quantity (the total vorticity) is preserved by (1.2), we see that $a_2(t)$ and $a_3(t)$ are in fact constant in time[1]. This implies that to the order given by (1.8), time-periodic wakes cannot be distinguished from stationary ones, though the actual value of the coefficients will differ from case to case. Without rigorous proof that the

[1] Note that it would be wrong to conclude that the drag and lift are constant in time from the fact that a_2 and a_3 are constant, as the volume integral of $\partial_t \mathbf{u}$ in (1.7) will generically no longer be zero for time-periodic flows. This is in agreement with results of numerical simulations, see e.g. [10, 12].

upstream asymptotics are as expected, we consider these physical interpretations as conjectures.

We end this section by noting that asymptotic results like (1.3) have been successfully used in numerical simulations of the stationary Navier–Stokes equations (1.2) in exterior domains, see [20, 21]. In particular, for fixed simulation domains, it allows to compute drag and lift coefficients with better accuracy than usual methods, while for fixed accuracy, smaller simulation domains can be used, thereby reducing significantly the CPU time needed for these computations. It is our hope that (1.8) could also be of such use in the time-periodic setting.

1.2. Reformulation of the problem

As in [9, 18], the starting point of the analysis is to write (1.1) as a dynamical system where x plays the role of time. To do so, we write $\mathbf{u} = \mathbf{u}_\infty + \mathbf{v}$ where $\mathbf{v} = (u, v)$ and introduce the vorticity $\omega = \partial_x v - \partial_y u$ and its derivative w.r.t. x as $\eta = \partial_x \omega$. Since the boundary data is assumed to be time-periodic, it is natural to assume that there is also a (discrete) Fourier decomposition of the various fields (this corresponds to the so-called *global mode* behavior, see also [13, 19, 22]) given by

$$u(x, y, t) = \sum_{n \in \mathbf{Z}} e^{i n \tau t} \, u_n(x, y), \tag{1.10}$$

with similar definitions for v, ω and η. In terms of this decomposition, the n-th Fourier coefficient of (1.1) reads (see also [9])

$$\begin{aligned}
\partial_x \omega_n &= \eta_n \\
\partial_x \eta_n &= \eta_n - \partial_y^2 \omega_n + i n \tau \omega_n + q_n \\
\partial_x u_n &= -\partial_y v_n \\
\partial_x v_n &= \partial_y u_n + \omega_n ,
\end{aligned} \tag{1.11}$$

with $q = u\eta + v\partial_y \omega$. Namely, the third equation in (1.11) is the incompressibility relation $\nabla \cdot \mathbf{u} = 0$, the last one is the definition of the vorticity, while substituting the first one in the second, one recovers the 'dynamical' part of (1.1). We interpret (1.11) as a new (hierarchy of) dynamical system(s) where the space variable x plays the role of time, $\partial_x \mathbf{z}_n = \mathcal{L}(\partial_y, n)\mathbf{z}_n + \mathbf{q}_n$ for $\mathbf{z} = (\omega, \eta, u, v)$ and $\mathbf{q} = (0, q, 0, 0)$.

Linear stability analysis applied to the continuous Fourier transform $y \to k$ of (1.11) immediately shows that this dynamical system is *ill posed*, as two of the four eigenvalues of $\mathcal{L}(-ik, n)$ have positive real part growing to infinity as $|k| \to \infty$ (see figure 1). In a linear setting (i.e. $q = 0$), we thus have to restrict the boundary conditions $\mathbf{z}(x_0)$ to the linear center-stable manifold, that is to those $\mathbf{z}(x_0)$ satisfying $\mathbf{z}(x_0) = \mathcal{P}_s \mathbf{z}(x_0)$ where $\mathcal{P}_s$ is the projection on the stable modes of $\mathcal{L}$. As we will see in Section 2, the linear center-stable manifold $\mathcal{P}_s \mathbf{z}_0$ is naturally

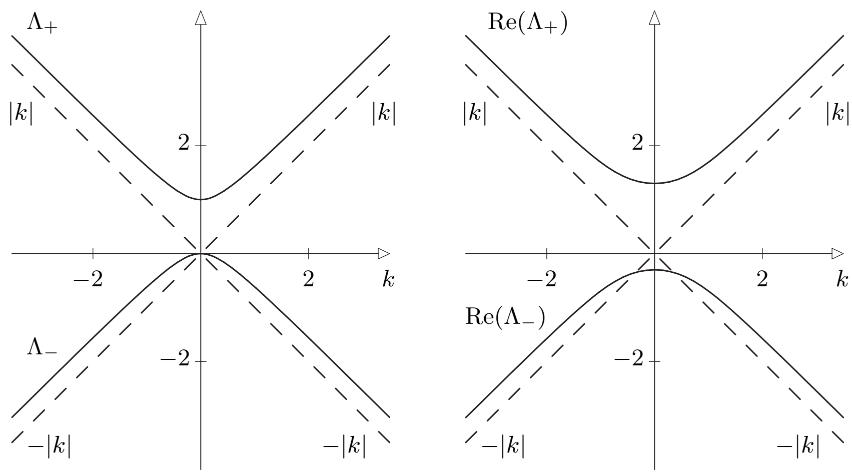

FIG. 1. (real part of the) eigenvalues of $\mathcal{L}(-ik,n)$ as a function of wavelength k, with $n\tau = 0$ in left panel, and $n\tau = 1$ in right panel.

parameterized by[2] $\mathcal{P}_s\mathbf{z}_0 = \mathcal{L}_V w + \mathcal{L}_E \nu$ where w is a 'vorticity-like' function (of y and t) and ν is a 'velocity-like' function (also of y and t). The mode $\mathcal{L}_V w$ is called 'vorticity mode' since the vorticity component[3] of $\mathcal{L}_V w$ is w. The mode $\mathcal{L}_E \nu$ (associated with the eigenvalue $-|k|$ of $\mathcal{L}(-ik)$) is called 'Eulerian' mode, since $\mathcal{L}_E\nu = (0,0,\nu,\mathcal{H}\nu)$ where $\mathcal{H}$ is the Hilbert transform on the boundary, and for well behaved ν, we can construct a stream function Ψ, harmonic in Ω_+, satisfying $\nabla\Psi|_{x=x_0} = (\nu,\mathcal{H}\nu)$. In the nonlinear setting, the boundary conditions have to be restricted to the nonlinear center-stable manifold. In Section 2, we will show that this manifold can be implicitly defined using Duhamel's variation of constants formula. Namely, we will cast (1.11) in an integral form given by

$$\begin{aligned}\mathbf{z}(x) = e^{\mathcal{L}(\partial_y)(x-x_0)}(\mathcal{L}_V w + \mathcal{L}_E\nu) + \int_{x_0}^{x} \mathrm{d}\tilde{x}\; e^{\mathcal{L}(\partial_y)(x-\tilde{x})}\mathcal{P}_s\mathbf{q}(\tilde{x}) \\ - \int_{x}^{\infty} \mathrm{d}\tilde{x}\; e^{\mathcal{L}(\partial_y)(x-\tilde{x})}(\mathbb{1}-\mathcal{P}_s)\mathbf{q}(\tilde{x}).\end{aligned} \tag{1.12}$$

Evaluation of (1.12) at $x = x_0$ then gives

$$\mathbf{z}(x_0) = \mathcal{L}_V w + \mathcal{L}_E\nu - \int_{x_0}^{\infty} \mathrm{d}\tilde{x}\; e^{\mathcal{L}(\partial_y)(x_0-\tilde{x})}(\mathbb{1}-\mathcal{P}_s)\mathbf{q}(\tilde{x}), \tag{1.13}$$

which accounts for the (small) nonlinear correction to the linear center-stable manifold.

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[2] Precise definitions of the operators $\mathcal{L}_V$ and $\mathcal{L}_E$ will be given in Section 2.

[3] A long wavelength expansion of $\mathcal{L}_V w$ gives $\mathcal{L}_V w \approx (w, \partial_y^2 w, -\mathcal{I}w, w)$ where $\mathcal{I} = \partial_y^{-1}$, see also (1.14).

Before stating our main results in a precise manner (see Subsection 1.4 below), we need the definition of some functional spaces and related norms. On an informal level, our results are twofold. We will use the integral formulation (1.12) to prove that if w and ν are in a certain class $\mathcal{C}_i$, there exist a (locally) unique solution of (1.1) in the Banach space $\mathcal{W}$ defined in the next section. We will then show that the asymptotic structure of these solutions is indeed given by (1.8) with $\varphi_0 > 0$. On the other hand, time-periodic solutions of (1.2) must satisfy (1.1) for all x_0 sufficiently large. We will then show that for solutions of (1.2) in a certain class $\mathcal{C}_u$, the $\mathbf{q}$ dependent terms in (1.12) are well defined, and thus solutions of (1.2) in $\mathcal{C}_u$ must also satisfy (1.12) for certain functions w and ν. The functions w and ν can be determined by inverting any two of the four (linear and local) relations (1.13), the two remaining relations, which correspond to the central-stable manifold condition in the dynamical system formulation (1.11), being automatically satisfied *since we know that the solution exist*. We will then show that the functions w and ν obtained in this way are in the class $\mathcal{C}_i$, which finally implies that solutions of (1.2) in $\mathcal{C}_u$ also satisfy (1.3) with $\varphi_0 > 0$.

1.3. Definitions

We now define the topology we will use to control the decompositions (1.10). Let $\langle x\rangle = \sqrt{1+x^2}$, $\rho_\beta(y) = |y|^\beta$ and $f(x,y,t) = \sum_{n\in\mathbf{Z}} e^{in\tau t} f_n(x,y)$ for $(x,y,t) \in [x_0,\infty)\times\mathbf{R}\times[0,\frac{2\pi}{\tau}]$. For $p\geq 1$, we define

$$\|\!|f|\!\|_{p,\sigma} = \sup_{x\geq x_0} \|f(x)\|_{p,\sigma}, \quad \|f(x)\|_{p,\sigma} = \langle x\rangle^\sigma \|f(x)\|_p = \langle x\rangle^\sigma \sum_{n\in\mathbf{Z}} \|f_n(x,\cdot)\|_{\mathrm{L}^p},$$
$$|f(x)|_p = \sup_{n\in\mathbf{Z}} \|f_n(x,\cdot)\|_{\mathrm{L}^p}, \qquad \|f\|_{p,\{\sigma_1,\sigma_2\}} = \sup_{x\geq 0} \langle x\rangle^{-\sigma_1} x^{\sigma_2} |f(x)|_p,$$

where $\|f_n(x,\cdot)\|_{\mathrm{L}^p}$ is the usual L^p-norm in the variable y and where we used the notation $\|f(x)\|_p$ as a shorthand to the more rigorous $\|f(x,\cdot)\|_p$. In the following, we will use repeatedly the operators $\mathcal{P}_0$, $\mathcal{P}$, $\mathcal{M}_n$, $\mathcal{I}$ and $\mathcal{S}$ defined by

$$\begin{aligned}
&\mathcal{P}_0 f = \frac{\tau}{2\pi}\int_0^{\frac{2\pi}{\tau}} \mathrm{d}t\ f(t), \quad \big(\mathcal{P}f\big)(t) = f(t) - \mathcal{P}_0 f, \quad \mathcal{M}_n(f) = \int_{\mathbf{R}} y^n f(y)\,\mathrm{d}y \\
&(\mathcal{I}f)(y) = \int_{-\infty}^{y} \mathrm{d}z\ \frac{f(z)}{2} - \int_y^\infty \mathrm{d}z\ \frac{f(z)}{2}, \quad (\mathcal{S}f)(y) = f(y) + f(-y).
\end{aligned} \tag{1.14}$$

Note that $\mathcal{I}$ is the (formal) inverse of ∂_y. We can now specify our basic functional space.

Definition 1.1. Let $\mathcal{C}_0^\infty = \big\{\{(u_n,v_n,\omega_n)\}_{n\in\mathbf{Z}}$ s.t. $(u_n,v_n,\omega_n) \in \mathcal{C}_0^\infty([x_0,\infty)\times\mathbf{R},\mathbf{R}^3)\ \forall n\in\mathbf{Z}\big\}$. We denote by $\mathcal{W}$ the Banach space obtained by closure of $\mathcal{C}_0^\infty$

under the norm

$$\begin{aligned}\|\!|(u,v,\omega)|\!\| = {} & \|\!|u|\!\|_{\infty,\frac{1}{2}} + \|\!|u|\!\|_{q,\frac{1}{2}-\frac{1}{q}} + \|\!|\partial_y u|\!\|_{r,1-\frac{1}{2r}-\eta} \\ & + \|\!|v|\!\|_{\infty,1-\varphi} + \|\!|v|\!\|_{p,1-\varphi-\frac{1}{p}} + \|\!|\partial_y v|\!\|_{r,\frac{3}{2}-\frac{1}{2r}-\xi} \\ & + \|\!|\omega|\!\|_{2,\frac{3}{4}} + \|\!|\rho_\beta\omega|\!\|_{2,\frac{3}{4}-\frac{\beta}{2}} + \|\!|\partial_y\omega|\!\|_{\infty,\frac{3}{2}} + \|\!|\partial_y\omega|\!\|_{1,1}.\end{aligned} \tag{1.15}$$

We will also use the notation $\|(u(x),v(x),\omega(x))\|$ to denote the natural norm on the trace space at x obtained by replacing all $\|\!|\cdot|\!\|_{p,\sigma}$ by $\|\cdot\|_{p,\sigma}$ in (1.15).

This choice is discussed at the end of this section. Note at this point that the 'expected' asymptotic decomposition (1.3) is in $\mathcal{W}$ if $p>1$. We now specify the class of solutions of (1.2) for which our results can be applied:

Definition 1.2. A solution (u,v,ω) of (1.2) is in the class $\mathcal{C}_u(\rho)$ if $\|\!|(u,v,\omega)|\!\| \leq \rho$ for some finite constant ρ and

$$\begin{aligned}&\tfrac{13}{7}\leq\beta\leq 3, \quad 1<p\leq q<2, \quad r>2, \quad 1-\tfrac{1}{p}<\varphi<\tfrac{r}{1+2r},\\ &0\leq\eta<\tfrac{1}{4}-\tfrac{\varphi}{2}, \quad \min(\varphi,\eta)\leq\xi\leq\tfrac{1}{2}, \quad \tfrac{1}{2}+\xi-\eta-2\varphi>0\\ &\tfrac{\langle\tau\rangle}{\tau}\leq\langle x_0\rangle^{\varphi}, \quad \tfrac{1}{2}+\eta-\xi-\tfrac{\varphi}{r}w>0.\end{aligned} \tag{1.16}$$

Our results are optimal if p and q are (very) close to 1 while η, ξ and φ are close to zero. To get bounds depending only on x_0 and not on the Strouhal number τ, we added the condition $\frac{\langle\tau\rangle}{\tau}\leq\langle x_0\rangle^{\varphi}$, which is only restrictive in the limit of vanishing Strouhal number. This is *not* expected to occur for time-periodic solutions of (1.2), if the Hopf bifurcation picture of [5, 14, 16, 17] is correct. We now define the class $\mathcal{C}_i$, consisting essentially of those functions w and ν for which the part of r.h.s. of (1.12) depending on w, ν is in $\mathcal{W}$ (see Lemma 3.1, 3.3 and 3.6):

Definition 1.3. We say that ν and w are in the class $\mathcal{C}_i(\rho)$ if $\|(\nu,\mathcal{H}\nu,w)\|\leq\rho$ for parameters satisfying (1.16) and $\mathcal{M}_0(\mathcal{P}_0 w)=0$.

Note that $\mathcal{M}_0(\mathcal{P}_0 w)$ is always well defined if $\|(0,0,w)\|<\infty$ since $\|w\|_{1,\frac{1}{2}}\leq\|(0,0,w)\|$ (see Lemma A.1). In the case of symmetric flows (i.e. u even in y and v odd in y), $\mathcal{M}_0(\mathcal{P}_0 w)=0$ is a trivial consequence of the fact that w is an odd function of y (it is also expected from (1.3) or (1.8)).

We end this section by making some comments on Definition 1.1. First, for the v component, we will need $\varphi>0$. Namely, as we will see, the optimal decay rate for v as $x\to\infty$ can only be obtained if $\mathcal{H}\nu\in\mathrm{L}^1$. But (apart from symmetric flows), $\mathcal{H}\nu(y)\sim 1/y$ as $y\to\infty$ (see (1.3)), so in general $\mathcal{H}\nu\notin\mathrm{L}^1$. The second comment is on the need of η and ξ. Basically, the problem is that $\partial_y u$ and $\partial_y v$ are naturally composed of sum of functions on two length scales ($y\sim\sqrt{x}$ and $y\sim x$, see e.g. (1.9)). Dependence on r of the decay exponents as $x\to\infty$ of L^r norms of such functions either vary like $1/(2r)$ for functions on the shorter scale or like $1/r$ for functions on the longer one. Our choice of exponents are thus 'wrong' on the scale $y \sim x$ and is 'corrected' by introducing η and ξ. These additional parameters would not be needed if we choose $r = \infty$, but in that case we would lose the boundedness of the Dirichlet–Neumann operator $\mathbf{v} \to \mathcal{H}\mathbf{v}$ in $\mathcal{W}$, which is needed to compare solutions of (1.1) and (1.2) (see Section 7).

1.4. Main results

We are now in position to state our results in a precise manner. The first one states that the topology of Definition 1.1 is well adapted to (1.1):

Theorem 1.4. *If x_0 is sufficiently large, and ν and w are in the class $\mathcal{C}_i(\rho)$ with parameters satisfying* (1.16)*, then there exist $\rho' > \rho$ and a (locally) unique solution to* (1.1) *in $\mathcal{C}_u(\rho')$ with parameters satisfying* (1.16)*.*

Once existence of solutions is proved, we give an intermediate result concerning the asymptotic properties of the vorticity and the first component of the velocity field:

Corollary 1.5. *Let $\mathbf{a}_1 = (-\mathcal{M}_0(\mathcal{I}\mathcal{P}_0 w) - \int_{\Omega_+} \mathcal{P}_0(v(x,y)\omega(x,y))\,\mathrm{d}x\mathrm{d}y, 0, 0, 0, 0, 0)$ and $u_{\mathbf{a}_1}$, $\omega_{\mathbf{a}_1}$ as in* (1.6) *and* (1.9)*, then for all $\varepsilon > 0$, solutions to* (1.1) *in $\mathcal{C}_u$ satisfy for all $\frac{1}{2} \leq \beta_0 \leq 1 - 2(1+\varepsilon)\varphi$ the estimates*

$$\begin{aligned} \|\omega - \omega_{\mathbf{a}_1}\|_{1,1-(1+\varepsilon)\varphi} + \|\omega - \omega_{\mathbf{a}_1}\|_{\infty,\frac{3}{2}-(1+\varepsilon)\varphi} &\leq C \\ \|\rho_{\beta_0}(\omega - \omega_{\mathbf{a}_1})\|_{2,\frac{5}{4}-\frac{\beta_0}{2}-(1+\varepsilon)\varphi} + \|u - u_{\mathbf{a}_1}\|_{\infty,1-(1+\varepsilon)\varphi} &\leq C, \end{aligned} \tag{1.17}$$

for some constant C which depends on x_0, $\|(u,v,\omega)\|$ and $\|(\nu,\mathcal{H}\nu,w)\|$.

Note that since $\varphi > 0$, in (1.3), the terms containing a_2 to a_6 are of smaller order than the remainder, which explains why these parameters are not yet specified at this point. We will then be able to get the complete asymptotic form in the

Corollary 1.6. *Assume that $\|\rho_{\frac{1}{2}}\mathbf{v}(x_0)\|_4 + \|\rho_{\frac{1}{2}-(1+\varepsilon)\varphi}\mathcal{S}\nu\|_1 + \|\rho_{\frac{1}{2}-(1+\varepsilon)\varphi}\mathcal{S}\mathcal{H}\nu\|_1 < \infty$ for some $(1+\varepsilon)\varphi < \frac{1}{8}$. Let a_1 denote the first component of $\mathbf{a}_1$ in Corollary 1.5, $a_2 = \mathcal{M}_0(\mathcal{S}\nu) - \int_{\Omega_+} \mathcal{P}_0(v(x,y)\omega(x,y))\,\mathrm{d}x\mathrm{d}y$ and $a_3 = \mathcal{M}_0(\mathcal{S}\mathcal{H}\nu)$, then there exists a constant a_4 such that for $\mathbf{a} = (a_1, a_2, a_3, a_4, a_1^2, a_1\mathcal{P}_0 a_3)$, $u_{\mathbf{a}}$, $v_{\mathbf{a}}$ and $\omega_{\mathbf{a}}$ as in* (1.6) *and* (1.9)*, solutions to* (1.1) *in $\mathcal{C}_u(\rho)$ satisfy for all $x \geq x_0$ and $\varepsilon > 0$, the estimate* (1.17) *and*

$$\|u - u_{\mathbf{a}}\|_{\infty,\frac{9}{8}-(1+\varepsilon)\varphi} + \|v - v_{\mathbf{a}}\|_{\infty,\frac{3}{2}-(1+\varepsilon)\varphi} \leq C \tag{1.18}$$

for some constant C which depends on x_0, $\|(u,v,\omega)\|$ and $\|(\nu,\mathcal{H}\nu,w)\|$.

As a first comment to this result, we want to note that we stopped at the stated asymptotic order in Corollary 1.6 for concision, but our method is constructive and could be systematically used to get higher order asymptotics. We also believe that (1.17) and (1.18) should hold with $\varphi = 0$ if $u_{\mathbf{a}}$, $v_{\mathbf{a}}$ and $\omega_{\mathbf{a}}$ contain appropriate logarithmic corrections. More interesting is the question of the actual decay rate of the first non-trivial time-periodic component. In our setting, we *cannot even exclude* that this decay rate is x^{-1}. Namely, as we motivated at the end of Section 1.1, we have *no rigorous evidence* but only strong physical motivations to believe that in the 'usual' problem (1.2) in an exterior domain, a_2 and a_3 are constant in time (in our setting, this follows if $\mathcal{M}_0(\mathcal{S}\nu)$ and $\mathcal{M}_0(\mathcal{S}\mathcal{H}\nu)$ are so). Though linear analysis (see also figure 1) indicates that the time harmonic of order $n \neq 0$ of the 'vorticity mode' associated with Λ_- should decay exponentially in the wake, this quantity is slaved to an inhomogeneous nonlinear term built from the $n = 0$ harmonic of the vorticity and the n^{th} harmonic of an 'Eulerian mode' (associated with the eigenvalue $-|k|$), both modes decaying only algebraically as $x \to \infty$. We thus conjecture that the first non-trivial time-periodic part will appear in the next (few) term(s) in the development.

It remains to show that the functional settings in which Corollary 1.6 is proved is reasonable in the sense that its conclusions are also true for the well known stationary solutions of the 'usual' exterior problem (1.2). To do so, we first note that

Proposition 1.7. *For any stationary solution of* (1.2) *"Physically Reasonable" (PR) in the sense of Finn and Smith* (*see e.g.* [6, 7, 8]), *the fields u, v and ω satisfy $\|\!|(u, v, \omega)|\!\| \leq C$ with parameters satisfying* (1.16) *if x_0 is sufficiently large. Furthermore $\|\rho_{\frac{1}{2}}\mathbf{v}(x_0)\|_4$ and $\|\rho_{\frac{1}{2}-(1+\varepsilon)\varphi}\mathcal{S}\mathbf{v}(x_0)\|_1$ are bounded for all $\varepsilon > 0$.*

and then conclude that

Theorem 1.8. *Assume that there exist a unique solution to* (1.1) *in $\mathcal{C}_u(\rho')$ with parameters satisfying* (1.16), *then if x_0 is sufficiently large, ν and w are in the class $\mathcal{C}_i(\rho)$ with parameters satisfying* (1.16). *If additionally $\|\rho_{\frac{1}{2}}\mathbf{v}(x_0)\|_4$ and $\|\rho_{\frac{1}{2}-(1+\varepsilon)\varphi}\mathcal{S}\mathbf{v}(x_0)\|_1$ are bounded, then for all $\varepsilon > 0$, it holds*

$$\|\rho_{\frac{1}{2}-(1+\varepsilon)\varphi}\mathcal{S}\nu\|_1 + \|\rho_{\frac{1}{2}-(1+\varepsilon)\varphi}\mathcal{S}\mathcal{H}\nu\|_1 \leq C_1(x_0, \|\!|(u, v, \omega)|\!\|). \tag{1.19}$$

From this theorem, Proposition 1.7 and the (local) uniqueness of the solutions in $\mathcal{C}_u$, we conclude that (PR) solutions satisfy the integral formulation (1.12) and the hypotheses of Corollary 1.6, which proves that its conclusions also hold for (PR) stationary solutions in the whole exterior domain.

1.5. Structure of the paper

Our first task in the remainder of this paper is to explicit the integral formulation (1.12). This is done in the next section (Section 2). The integral formulation is then used in Section 3 to prove Theorem 1.4, in Section 4 to prove Corollary 1.5 and in Section 5 to prove Corollary 1.6. The proof of Proposition 1.7 is delayed until Section 6, while that of Theorem 1.8 is delayed until Section 7.

2. Integral formulation

We now derive an explicit formula for the integral formulation (1.12) of the solution of (1.1) and (1.11). All the material of this section is very similar to [9, 18] where the case $\tau = 0$ was treated. For completeness, we now reproduce some of the analysis here, encompassing the additional term proportional to the Strouhal number τ.

For further reference, we first note that the representation (1.11) with $q = u\eta + v\partial_y\omega$ is not the only possibility. Namely, using the incompressibility relation $\partial_x u = -\partial_y v$ and the definition of the vorticity, we may cast the nonlinearity q in the following equivalent and more useful forms

$$q = \partial_x(u\omega) + \partial_y(v\omega) \equiv \partial_x(P) + \partial_y(Q) = (\partial_x^2 + \partial_y^2)R + 2\partial_y Q,$$

since $P = u\omega = \partial_x R + \partial_y S$ and $Q = v\omega = -\partial_y R + \partial_x S$ where $R = uv$, $S = \frac{1}{2}(v^2 - u^2)$. We then note that performing a (continuous) Fourier transform[4] $f(k) = \int_{\mathbf{R}} e^{iky} f(y)$ leads for each $n \in \mathbf{Z}$ to a system of the form $\partial_x \mathbf{z}_n = \mathcal{L}(-ik,n)\mathbf{z}_n + \mathbf{q}_n$, with $\mathbf{z} = (\omega, \eta, u, v)$, $\mathbf{q} = (0, q, 0, 0)$. As in [9], the matrix $\mathcal{L}(-ik,n)$ can be diagonalized. Namely, define $\sigma(k) = \mathrm{sign}(k)$, $\Lambda_0 = \sqrt{1 + 4(k^2 + in\tau)}$ and $\Lambda_\pm = \frac{1 \pm \Lambda_0}{2}$, and set $\mathbf{z} = \mathcal{S}^{-1}\mathbf{y}$, with $\mathbf{y} = (\omega_+, \omega_-, u_+, u_-)$ and

$$\mathcal{S}(-ik,n) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ \Lambda_+ & \Lambda_- & 0 & 0 \\ \frac{ik}{\Lambda_+ + in\tau} & \frac{ik}{\Lambda_- + in\tau} & 1 & 1 \\ \frac{\Lambda_+}{\Lambda_+ + in\tau} & \frac{\Lambda_-}{\Lambda_- + in\tau} & -i\sigma & i\sigma \end{pmatrix},$$

then we get $\mathcal{S}(-ik,n)^{-1}\mathcal{L}(-ik,n)\mathcal{S}(-ik,n) = \mathrm{diag}(\Lambda_+, \Lambda_-, |k|, -|k|)$ (see figure 1 for a graphical display of the dispersion relations). The symbols of the operators $\mathcal{L}_V$, resp. $\mathcal{L}_E$ used in Subsection 1.2 to characterize the linear center-stable manifold are the second, resp. fourth column of $\mathcal{S}$, since the two equations corresponding to the '+' modes are linearly unstable (the real part of Λ_+ is positive). Integrating the unstable modes backwards from $x = \infty$, where we set them to 0 (see also [18]), we get

$$\omega_+(x) = -\int_x^\infty \mathrm{d}\tilde{x} \frac{e^{\Lambda_+(x-\tilde{x})}}{\Lambda_0} q(\tilde{x})$$

$$\omega_-(x) = e^{\Lambda_-(x-x_0)}\tilde{\omega}_0 - \int_{x_0}^x \mathrm{d}\tilde{x}\, \frac{e^{\Lambda_-(x-\tilde{x})}}{\Lambda_0} q(\tilde{x})$$

[4] We distinguish functions and their Fourier transform only from their arguments 'k', resp. 'y' in Fourier resp. direct space.

$$u_+(x) = -\frac{1}{2}\int_x^\infty \mathrm{d}\tilde{x}\ \frac{e^{|k|(x-\tilde{x})}}{ik-n\tau\sigma}q(\tilde{x})$$
$$u_-(x,k) = e^{-|k|(x-x_0)}\tilde{u}_0 + \frac{1}{2}\int_{x_0}^x \mathrm{d}\tilde{x}\ \frac{e^{-|k|(x-\tilde{x})}}{ik+n\tau\sigma}q(\tilde{x}),$$

for some functions $\tilde{\omega}_0$ and $\tilde{u}_0$ to be specified. Integrating by parts the integrals involving $\partial_x P$ in $\omega_\pm$, replacing $q = (\partial_x^2+\partial_y^2)R + 2\partial_y Q$ in $u_\pm$ and integrating twice by parts the term involving $\partial_x^2 R$, we find

$$\omega_+(x) = \frac{P(x)}{\Lambda_0} - \int_x^\infty \mathrm{d}\tilde{x}\ \frac{e^{\Lambda_+(x-\tilde{x})}}{\Lambda_0}q_+(\tilde{x})$$
$$\omega_-(x) = e^{\Lambda_-(x-x_0)}w - \frac{P(x)}{\Lambda_0} - \int_{x_0}^x \mathrm{d}\tilde{x}\ \frac{e^{\Lambda_-(x-\tilde{x})}}{\Lambda_0}q_-(\tilde{x})$$
$$u_+(x) = \frac{P(x)+ikS(x)+|k|R(x)}{2(ik-n\tau\sigma)} + \int_x^\infty \mathrm{d}\tilde{x}\ \frac{ike^{|k|(x-\tilde{x})}}{ik-n\tau\sigma}Q(\tilde{x})$$
$$u_-(x) = e^{-|k|(x-x_0)}\nu + \frac{P(x)+ikS(x)-|k|R(x)}{2(ik+n\tau\sigma)} - \int_{x_0}^x \mathrm{d}\tilde{x}\ \frac{ike^{-|k|(x-\tilde{x})}}{ik+n\tau\sigma}Q(\tilde{x}),$$

where $q_\pm = \Lambda_\pm P - ikQ$, $\nu(k) = \tilde{u}_0(k) - \frac{P(x_0)+ikB(x_0)-|k|A(x_0)}{2(ik+n\tau\sigma)}$ and $w(k) = \tilde{\omega}_0(k) + \frac{P(x_0)}{\Lambda_0}$. Then, a little algebra shows that when reconstructing ω, u and v, the terms involving $P(x)$ cancel out exactly. We thus find, after inverse Fourier transform

$$\begin{aligned} u(x) &= u_L(x) + u_N(x), \qquad & v(x) &= v_L(x) + v_N(x) \\ \omega(x) &= \omega_L(x) + \omega_N(x), \end{aligned} \tag{2.1}$$

with $u_L(x) = \sum_{i=1}^2 u_{L,i}(x)$, $v_L(x) = \sum_{i=1}^3 v_{L,i}(x)$, $u_N(x) = \sum_{i=1}^6 u_{N,i}(x)$, $v_N(x) = \sum_{i=1}^8 v_{N,i}(x)$ and $\omega_N(x) = \sum_{i=1}^4 \omega_{N,i}(x)$, where $v_{N,7}(x) = \omega_{N,1}(x) + \omega_{N,2}(x) + \omega_{N,3}(x)$, $v_{N,8}(x) = \omega_{N,4}(x)$, and

$$\begin{aligned} u_{L,1}(x) &= K_1(x-x_0)\mathcal{L}_u w, \qquad & v_{L,1}(x) &= K_1(x-x_0)(\mathcal{L}_v - \mathbb{1})w, \\ \omega_L(x) &= K_1(x-x_0)w, & u_{L,2}(x) &= K_0(x-x_0)\nu, \\ v_{L,2}(x) &= K_0(x-x_0)\mathcal{H}\nu, & v_{L,3}(x) &= \omega_L(x), \end{aligned}$$

$$\begin{aligned} u_{N,1}(x) &= -\mathcal{J}_1[K_8,Q](x), & u_{N,2}(x) &= -\mathcal{J}_1[K_{12},Q](x), \\ u_{N,3}(x) &= \mathcal{J}_1[K_2-K_9,P](x), & u_{N,4}(x) &= \mathcal{J}_2[K_{12}^*,Q](x), \end{aligned} \tag{2.2}$$
$$\begin{aligned} v_{N,1}(x) &= \mathcal{J}_1[K_9,Q](x), & v_{N,2}(x) &= \mathcal{J}_1[K_{13},Q](x), \\ v_{N,3}(x) &= \mathcal{J}_1[K_{10}+K_{11},P](x), & v_{N,4}(x) &= -\mathcal{J}_2[K_{13}^*,Q](x), \end{aligned} \tag{2.3}$$
$$\begin{aligned} u_{N,5}(x) &= \mathcal{J}_2[\mathcal{E}(K_2-K_4),P](x) - \mathcal{J}_2[\mathcal{E}K_3,Q](x), \\ v_{N,5}(x) &= -\mathcal{J}_2[\mathcal{E}K_5,P](x) + \mathcal{J}_2[\mathcal{E}K_4,Q](x), \end{aligned}$$
$$u_{N,6}(x) = \mathcal{L}_1 S(x) - \mathcal{L}_2 R(x), \qquad v_{N,6}(x) = -\mathcal{L}_1 R(x) - \mathcal{L}_2 S(x),$$

$$\omega_{N,1}(x) = -\mathcal{J}_1[K_2,Q](x), \qquad \omega_{N,2}(x) = -\mathcal{J}_1[K_6,P](x),$$

$$\omega_{N,3}(x) = -\mathcal{J}_1[K_7, P](x),$$
$$\omega_{N,4}(x) = -\mathcal{J}_2[\mathcal{E}(K_1 + K_6 + K_7), P](x) - \mathcal{J}_2[\mathcal{E}K_2, Q](x),$$

where

$$\mathcal{J}_1[K, f](x) = \int_{x_0}^{x} \mathrm{d}\tilde{x}\; K(x - \tilde{x})\, f(\tilde{x}), \qquad \mathcal{J}_2[K, f](x) = \int_{x}^{\infty} \mathrm{d}\tilde{x}\; K(\tilde{x} - x)\, f(\tilde{x})$$

and, in terms of their symbols,

$$\mathcal{L}_1 = \frac{k^2}{k^2 + (n\tau)^2},\ \mathcal{L}_2 = \frac{|k|n\tau}{k^2 + (n\tau)^2},\ \mathcal{L}_u = (\Lambda_- + in\tau)^{-1} ik,\ \mathcal{L}_v = (\Lambda_- + in\tau)^{-1}\Lambda_-,$$

and for $i = 0, \ldots, 13$, $K_i(x)$ is the convolution operator with the inverse Fourier transform of $K_i(x,k)$ and $(\mathcal{E}K_i)(x)$ with that of $e^{-x}K_i(x,k)$, where

$$\begin{array}{lll}
K_0(x,k) = e^{-|k|x} & K_1(x,k) = e^{\Lambda_- x} & K_2(x,k) = -\frac{ike^{\Lambda_- x}}{\Lambda_0} \\
K_3(x,k) = \frac{k^2 e^{\Lambda_- x}}{\Lambda_0(\Lambda_+ + in\tau)} & K_4(x,k) = \frac{kn\tau e^{\Lambda_- x}}{\Lambda_0(\Lambda_+ + in\tau)} & K_5(x,k) = -\frac{in\tau\Lambda_+ e^{\Lambda_- x}}{\Lambda_0(\Lambda_+ + in\tau)} \\
K_6(x,k) = \frac{\mathrm{Re}(\Lambda_-)}{\Lambda_0} e^{\Lambda_- x} & K_7(x,k) = \frac{i\mathrm{Im}(\Lambda_-)}{\Lambda_0} e^{\Lambda_- x} & K_8(x,k) = \frac{k^2 e^{\Lambda_- x}}{\Lambda_0(\Lambda_- + in\tau)} \\
K_9(x,k) = \frac{kn\tau e^{\Lambda_- x}}{\Lambda_0(\Lambda_- + in\tau)} & K_{10}(x,k) = \frac{in\tau\mathrm{Re}(\Lambda_-) e^{\Lambda_- x}}{\Lambda_0(\Lambda_- + in\tau)} & K_{11}(x,k) = \frac{-n\tau\mathrm{Im}(\Lambda_-) e^{\Lambda_- x}}{\Lambda_0(\Lambda_- + in\tau)} \\
K_{12}(x,k) = \frac{ike^{-|k|x}}{ik + n\tau\sigma} & K_{13}(x,k) = \frac{|k|e^{-|k|x}}{ik + n\tau\sigma}\,. &
\end{array}$$

Various estimates of these kernels are given in Appendix A. Intuitively, the two kernels K_{12} and K_{13} behave like Poisson's kernels $\frac{1}{\pi}\frac{x}{x^2+y^2}$ and $\frac{1}{\pi}\frac{y}{x^2+y^2}$, while all the other kernels behave like y derivatives or primitives of K_1 according to the expansion of their pre-factor as $|k| \to 0$ or $|k| \to \infty$. We thus need to understand the basic properties of K_1. To do so, we define

$$b(\alpha) = \tfrac{1}{4}\Big(1 - \sqrt{\tfrac{1+\sqrt{1+16\alpha^2}}{2}}\Big)\ , \quad c(\alpha) = \tfrac{1}{2}\sqrt{\tfrac{1+\sqrt{1+16\alpha^2}}{2+32\alpha^2}},$$

and note that (see also figure 1 on page 300)

$$\mathrm{Re}(\Lambda_-) \le \begin{cases} b(n\tau) - c(n\tau)k^2 & \forall |k| \le 1 \\ b(n\tau) - \frac{|k|}{2} & \forall |k| \ge 1 \end{cases} \quad \text{and} \quad \left|\frac{1}{\Lambda_0}\right| \le \begin{cases} (1 + (n\tau)^2)^{-1/4} \\ (1 + k^2)^{-1/2} \end{cases}.$$

The kernel K_1 corresponding to $e^{\Lambda_- x}$ thus behaves like a superposition of a kernel of Poisson's type with a heat kernel (see also Lemma A.10):

$$K_1(x,y) \approx e^{b(n\tau)x}\Big(\frac{e^{-\frac{y^2}{4c(n\tau)x}}}{\sqrt{4\pi c(n\tau)\,x}} + \tfrac{1}{\pi}\tfrac{2x}{x^2+4y^2}\Big). \tag{2.4}$$

Most results of Appendix A can be easily derived from this analogy, in particular we see in (2.4) that $\partial_y^m K_1 \sim x^{-m}\langle x\rangle^{\frac{m}{2}} K_1$ and that since $b(0) = 0$ and $b(\tau) < 0$, L^p estimates on K_1 will decay *at most algebraically* as $x \to \infty$, while the same estimates on $\mathcal{P}K_1$ will decay *exponentially faster*.

Unfortunately, without using (yet unknown) compensations between the various terms in (2.1), we cannot conclude from this last remark that e.g. $\mathcal{P}\omega$ will decay exponentially as $x \to \infty$. Namely, we first note that e.g. $\mathcal{P}(\omega_{N,2}(x) + \omega_{N,3}(x)) \approx \mathcal{P}P(x)$ as the integral defining $\mathcal{P}(\omega_{N,2} + \omega_{N,3})$ is dominated by the contribution of the region $\tilde{x} \approx x$. Then, we note that $\mathcal{P}P(x) = \big(\mathcal{P}_0\omega(x)\big)\big(\mathcal{P}u(x)\big) + \ldots$, and that both $\mathcal{P}_0\omega$ and $\mathcal{P}u$ decay at most algebraically because $\mathcal{P}_0K_1$ and $\mathcal{P}u_{L,2}$ do so (unless $\mathcal{P}\nu = 0$, in which case a more refined analysis shows that e.g. $\mathcal{P}u_{N,2}$ necessarily decays algebraically).

3. 'Evolution' estimates and the proof of Theorem 1.4

Our next task is to prove Theorem 1.4, which states that for each boundary data in $\mathcal{C}_i$, there exists in $\mathcal{C}_u$ a (locally) unique solution to (2.1). The proof follows easily from the contraction mapping principle. For fixed ν and w in $\mathcal{C}_i$, we define the map $\mathcal{F} : \mathcal{W} \to \mathcal{W}$ by $\mathcal{F}(\mathbf{v}, \omega) =$ r.h.s. of (2.1). In the remainder of this section, we will prove that if the parameters satisfy (1.16), then there exists $\kappa > 0$, such that for all u_i, v_i and ω_i, $i = 1, 2$ with $|||(u_i, v_i, \omega_i)||| \leq \tilde{\rho}$, we have

$$|||\mathcal{F}(u_i, v_i, \omega_i)||| \leq C_1\|(\nu, \mathcal{H}\nu, w)\| + C_2\langle x_0\rangle^{-\kappa}\tilde{\rho}^2,$$
$$|||\mathcal{F}(u_1, v_1, \omega_1) - \mathcal{F}(u_2, v_2, \omega_2)||| \leq C_2\langle x_0\rangle^{-\kappa}|||(u_1 - u_2, v_1 - v_2, \omega_1 - \omega_2)|||\ \tilde{\rho}.$$

Let $\rho > 0$ and $0 < \varepsilon < \frac{1}{2}$. If $\|(\nu, \mathcal{H}\nu, w)\| \leq \rho$ and $\langle x_0\rangle > \big(C_1C_2\rho\varepsilon^{-1}\big)^{\frac{1}{\kappa}}$, the map $\mathcal{F}$ is a contraction in $\mathcal{B}_0((1+\varepsilon)C_1\rho) \subset \mathcal{W}$. By classical arguments, the approximating sequence $(u_{n+1}, v_{n+1}, \omega_{n+1}) = \mathcal{F}(u_n, v_n, \omega_n)$ for $n > 1$ and $(u_1, v_1, \omega_1) = \mathcal{F}(0, 0)$ converges to the unique solution of (1.1) in $\mathcal{B}_0((1 + \varepsilon)C_1\rho) \subset \mathcal{W}$. The proof of Theorem 1.4 is thus completed, provided we prove the above estimates on $\mathcal{F}$. This part of the proof is split between the next subsections as follows: Subsection 3.2 is devoted to the terms $u_{L,i}$, $v_{L,i}$ and ω_L and Subsection 3.3 to the terms $u_{N,i}$, $v_{N,i}$ and $\omega_{N,i}$. In the remainder of this section, the letter C stands for a constant which may change its value from instance to instance, but is independent of x_0, $|||(u, v, \omega)|||$ and $\|(\nu, \mathcal{H}\nu, w)\|$.

3.1. Preliminaries

In this whole section, we will use that for K a convolution kernel (in the variable y) acting on a function f (of y) and $\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2} = 1 + \frac{1}{s}$, we have (see Subsection 1.3 for the definitions of the norms)

$$\|\rho_\beta K f\|_2 \leq |\rho_\beta K|_2\|f\|_1 + |K|_1\|\rho_\beta f\|_2, \tag{3.1}$$
$$\|K f\|_s \leq \min\Big(|K|_{p_1}\|f\|_{q_1},\ |K|_{p_2}\|f\|_{q_2}\Big). \tag{3.2}$$

We will also use variants of (3.1) and (3.2) following from $Kf = \partial_y K\mathcal{I}f$ and $\partial_y(Kf) = (\partial_y K)f = K(\partial_y f)$. In particular, (3.1)–(3.2) and their variants give a great freedom for the way we will actually do the estimates. Our main concern and difficulty will be to get optimal decay rates as $x \to \infty$. As a rule (particularly in Subsection 3.3), we will choose p_1 as small as possible to cover regions where the x argument of K is small and p_2 as large as possible in regions where that argument is large. For concision, we will often omit the arguments in the K's and f's when no confusion is possible. For the same reason, we will use (3.1)–(3.2) without reference or even sometimes without explicit statement of the choice made for the parameters.

We also note for further reference that using $\|f\|_\infty \le (\|f\|_2 \|\partial_y f\|_2)^{\frac{1}{2}}$, the interpolation inequality, $0 < \varphi < \frac{1}{2}$ and $\frac{1}{2} + \eta - \xi \ge 0$, we have for some constant C that

$$\|(f,0,0)\| \le C\|(0,f,0)\| \le C\|(0,0,f)\|, \tag{3.3}$$

and that the nonlinearities R, S, P and Q satisfy

$$\begin{aligned} \|P\|_{m,\frac{3}{2}-\frac{1}{2m}} + \|\partial_y P\|_{n,2-\frac{1}{2n}-\eta} + \|\rho_\beta P\|_{2,\frac{5}{4}-\frac{\beta}{2}} &\le C\|(u,v,\omega)\|^2, \\ \|Q\|_{m,2-\varphi-\frac{1}{2m}} + \|\rho_\beta Q\|_{2,\frac{7}{4}-\varphi-\frac{\beta}{2}} + \|\partial_y Q\|_{n,\frac{5}{2}-\frac{1}{2n}-\xi} &\le C\|(u,v,\omega)\|^2, \\ \|R\|_{m,\frac{3}{2}-\varphi-\frac{1}{m}} + \|\partial_y R\|_{r,\frac{3}{2}-\eta-\frac{1}{2r}} + &\le C\|(u,v,\omega)\|^2, \\ \|S\|_{m,1-\frac{1}{m}} + \|\partial_y S\|_{r,\frac{3}{2}-\eta-\frac{1}{2r}} &\le C\|(u,v,\omega)\|^2, \end{aligned} \tag{3.4}$$

for all $1 \le m \le \infty$ and $1 \le n \le r$. To establish (3.4), we used $\|\omega\|_{\infty,1} + \|\omega\|_{1,\frac{1}{2}} \le \|(0,0,\omega)\|$, since $\|\omega\|_1 \le C_\beta \|\omega\|_2^{1-\frac{1}{2\beta}} \|\rho_\beta\omega\|_2^{\frac{1}{2\beta}}$ and $\|\omega\|_\infty^2 \le \|\omega\|_2 \|\partial_y\omega\|_2$, see also Lemma A.1.

3.2. The 'linear' terms

In this subsection, we prove that $\|(u_L, v_L, \omega_L)\| \le C\|(\nu, \mathcal{H}\nu, w)\|$ provided ν and w are in the Class $\mathcal{C}_i$ of Definition 1.3. By (3.3), it will be sufficient to prove that $\|(u_{L,1}+u_{L,2}, v_{L,1}+v_{L,2}, \omega_L)\| \le C\|(\nu, \mathcal{H}\nu, w)\|$. For convenience, this inequality is split component-wise in three following Lemmas. The general idea of the proofs is to consider separately the regions $x_0 \le x \le 2x_0$ and $x \ge 2x_0$. In the first region, we will use the fact that $|K_0(x-x_0)|_1 + |K_1(x-x_0)|_1$ is uniformly bounded (thus $K_0\cdot$ and $K_1\cdot$ are L^p-bounded operators for all $p \ge 1$), whereas in the region $x \ge 2x_0$, we will essentially use that $|K_0(x-x_0)|_p + |K_1(x-x_0)|_p$ decays as $x \to \infty$ as soon as $p > 1$.

Lemma 3.1. *If* (1.16) *holds, then there exists a constant C such that* $\|(0,0,\omega_L)\| \le C\|(0,0,w)\|$.

Proof. We first note that since $|K_1|_1 \le C$, we have

$$\|\omega_L\|_{2,\frac{3}{4}} \le C \sup_{x_0 \le x \le 2x_0} \langle x\rangle^{\frac{3}{4}} \|w\|_2 + \sup_{x \ge 2x_0} \langle x\rangle^{\frac{3}{4}} \big(|\partial_y K_1|_2 \|\mathcal{P}_0 \mathcal{I} w\|_1 + |\mathcal{P} K_1|_2 \|w\|_1\big)$$

$$\leq C(\|w\|_{2,\frac{3}{4}} + \|w\|_{1,\frac{1}{2}} + \|\mathcal{P}_0\mathcal{I}w\|_1),$$

where we used Lemma A.5, that $x - x_0 \geq \frac{x}{2}$ if $x \geq 2x_0$, and that $\langle x\rangle^{\frac{1}{2}} e^{b(\tau)x} \leq \frac{\langle\tau\rangle}{\tau} \leq \langle x_0\rangle^{\frac{1}{2}}$. Omitting the details for concision, we get by the same arguments that

$$\|\!|\partial_y \omega_L|\!\|_{\infty,\frac{3}{2}} + \|\!|\partial_y \omega_L|\!\|_{1,1} \leq C(\|\partial_y w\|_{\infty,\frac{3}{2}} + \|\partial_y w\|_{1,1} + \|w\|_{1,\frac{1}{2}} + \|\mathcal{P}_0\mathcal{I}w\|_1).$$

Next, we note that for all $z \in \mathbf{R}$, we can write $\rho_\beta(y) = \rho_\beta(y-z) + L(y,z)$ with $|L(y,z)| \leq C(\rho_\beta(z) + \rho_1(z)\rho_{\beta-1}(y-z))$, so that (here we use that $\beta > \frac{3}{2}$)

$$\begin{aligned}
\|\!|\rho_\beta \mathcal{P}\omega_L|\!\|_{2,\frac{3}{4}-\frac{\beta}{2}} &\leq C\|\rho_\beta w\|_{2,\frac{3}{4}-\frac{\beta}{2}} + C\sup_{x\geq x_0}\langle x\rangle^{\frac{3}{4}-\frac{\beta}{2}}|P\rho_\beta K_1|_2\|w\|_1 \\
&\leq C\|\rho_\beta w\|_{2,\frac{3}{4}-\frac{\beta}{2}} + C\|w\|_{1,\frac{1}{2}}, \\
\|\!|\rho_\beta \mathcal{P}_0\omega_L|\!\|_{2,\frac{3}{4}-\frac{\beta}{2}} &\leq C\sup_{x\geq x_0}\langle x\rangle^{\frac{3}{4}-\frac{\beta}{2}}\Big(\|(\rho_\beta K_1)\mathcal{P}_0 w\|_2 + \|\rho_\beta w\|_2 + |\rho_{\beta-1}K_1|_2\|\rho_1 w\|_1\Big) \\
&\leq C\Big(\sup_{x\geq x_0}\langle x\rangle^{\frac{3}{4}-\frac{\beta}{2}}|\partial_y(\rho_\beta K_1)|_2\|\mathcal{I}\mathcal{P}_0 w\|_1\Big) + C\|(0,0,w)\|,
\end{aligned}$$

where we used $\|(\rho_\beta K_1)\mathcal{P}_0 w\|_2 = \|(\partial_y \rho_\beta K_1)\mathcal{I}\mathcal{P}_0 w\|_2$ and that since $\beta > \frac{3}{2}$, we have

$$\|\rho_1 w\|_{1,\frac{3}{4}-\frac{\beta}{2}} \leq \|w\|_{2,\frac{3}{4}-\frac{\beta}{2}} + \|\rho_\beta w\|_{2,\frac{3}{4}-\frac{\beta}{2}} \leq \|(0,0,w)\|. \tag{3.5}$$

The proof is completed using $\|w\|_{1,\frac{1}{2}} \leq \|(0,0,w)\|$ and Lemma 3.2 below. $\square$

Lemma 3.2. *Let $\beta > \frac{3}{2}$ and $0 \leq \gamma < \beta - \frac{3}{2}$, $\mathcal{Z}_\beta = \{\|(1+\rho_\beta)f\|_2 < \infty$ and $\mathcal{M}_0(f) = \int_{\mathbf{R}} f(y)\mathrm{d}y = 0\}$. Then there exist constants $C_\beta, C_{\beta,\gamma}$ such that for all $f \in \mathcal{Z}_\beta$,*

$$\|\mathcal{I}f\|_\infty \leq C_\beta \|f\|_2^{1-\frac{1}{2\beta}}\|\rho_\beta f\|_2^{\frac{1}{2\beta}} \quad \text{and} \quad \|\rho_\gamma \mathcal{I}f\|_1 \leq C_{\beta,\gamma}\|f\|_2^{1-\frac{3}{2\beta}-\frac{\gamma}{\beta}}\|\rho_\beta f\|_2^{\frac{3}{2\beta}+\frac{\gamma}{\beta}}.$$

The first inequality is also valid if $\mathcal{M}_0(f) \neq 0$.

Proof. Let $\beta > \frac{3}{2}$ and $a > 0$. Since $\|\mathcal{I}f\|_\infty \leq \|f\|_1$, the first inequality follows from Lemma A.1. Then, since $\mathcal{M}_0(f) = 0$, we have $\mathcal{I}f(y) = \int_y^{\mathrm{sign}(y)\infty}\mathrm{d}z\, f(z)$, so that the proof follows easily from

$$|\mathcal{I}f(y)|^2 \leq \Big(a\|f\|_2 + \|\rho_\beta f\|_2\Big)^2 \int_{|y|}^\infty \frac{\mathrm{d}z}{(a+|z|^\beta)^2},$$

the fact that $\int_0^\infty \mathrm{d}y\ |y|^\gamma \left(\int_y^\infty \frac{\mathrm{d}z}{(1+|z|^\beta)^2}\right)^{\frac{1}{2}} < \infty$ if $\gamma < \beta - \frac{3}{2}$ and minimizing the bound as a function of a. $\square$

Lemma 3.3. *If* (1.16) *holds, then there exists a constant C such that*

$$\|\!|(u_{L,1}, v_{L,1}, 0)|\!\| \leq C\|(0,0,w)\|.$$

Proof. We first note that $\|u_{L,1}\|_{q,\frac{1}{2}-\frac{1}{2q}} \leq \|u_{L,1}\|_1 + \|u_{L,1}\|_{\infty,\frac{1}{2}}$ and $\|v_{L,1}\|_{p,1-\frac{1}{2p}-\varphi} \leq \|v_{L,1}\|_{1,\frac{1}{2}-\varphi} + \|v_{L,1}\|_{\infty,1-\varphi}$. Let $\tilde{\mathcal{L}}_v = \mathcal{L}_v - \mathbb{1}$. We then note that $\tilde{\mathcal{L}}_v = \frac{-in\tau}{\Lambda_- + in\tau}$, in particular, $\mathcal{P}\tilde{\mathcal{L}}_v = \tilde{\mathcal{L}}_v$ and $\mathcal{P}e^{cb(n\tau)x} \leq e^{cb(\tau)x}$ for all $c > 0$. As in Lemma 3.1, since $x - x_0 \geq \frac{x}{2}$ for $x \geq 2x_0$, $b(\tau) < 0$, $\langle x \rangle e^{\frac{b(\tau)x}{8}} \leq \frac{\langle \tau \rangle}{\tau} \leq \langle x_0 \rangle^{\frac{1}{2}}$ and $\xi \geq \varphi$, we have that

$$
\begin{aligned}
\|v_{L,1}\|_{1,\frac{1}{2}-\varphi} + &\leq C(\|\tilde{\mathcal{L}}_v w\|_{1,\frac{1}{2}-\varphi} + \sup_{x \geq 2x_0} \langle x \rangle^{\frac{1}{2}-\varphi} |\mathcal{P}K_1|_1 \|\tilde{\mathcal{L}}_v w\|_1 \leq C\|\tilde{\mathcal{L}}_v w\|_{1,\frac{1}{2}-\varphi}, \\
\|v_{L,1}\|_{\infty,1-\varphi} &\leq C\|\tilde{\mathcal{L}}_v w\|_{\infty,1-\varphi} + \sup_{x \geq 2x_0} \langle x \rangle^{1-\frac{1}{2s}-\varphi} |\mathcal{P}K_1|_s \|\tilde{\mathcal{L}}_v w\|_1, \\
&\leq C(\|\tilde{\mathcal{L}}_v w\|_{1,\frac{1}{2}-\varphi} + \|\tilde{\mathcal{L}}_v w\|_{\infty,1-\varphi}),
\end{aligned}
$$

$$
\begin{aligned}
\|\partial_y v_{L,1}\|_{r,\frac{3}{2}-\frac{1}{2r}-\xi} &\leq \|\partial_y \tilde{\mathcal{L}}_v w\|_{r,\frac{3}{2}-\frac{1}{2r}-\xi} + C \sup_{x \geq 2x_0} \langle x \rangle^{\frac{3}{2}-\frac{1}{2r}-\xi} |\mathcal{P}\partial_y K_1|_r \|\tilde{\mathcal{L}}_v w\|_1, \\
&\leq C\Big(\|\partial_y \tilde{\mathcal{L}}_v \omega\|_{r,\frac{3}{2}-\frac{1}{2r}-\xi} + \|\tilde{\mathcal{L}}_v \omega\|_{1,\frac{1}{2}-\varphi}\Big).
\end{aligned}
$$

Then, proceeding as above, we get

$$
\|(u_{L,1}, 0, 0)\| \leq C\big(\|\mathcal{L}_u w\|_1 + \|\mathcal{L}_u w\|_{\infty,\frac{1}{2}} + \|\partial_y \mathcal{L}_u w\|_{r,1-\frac{1}{2r}-\eta}\big).
$$

The proof is then completed using Lemma 3.4 and 3.5 below. $\square$

Lemma 3.4 (Mikhlin–Hörmander). *Let $m : \mathbf{R} \to \mathbf{C}$, and assume that the following quantities are bounded: $m_0 = \sup_{k \in \mathbf{R}} |m(k)| + |k\partial_k m(k)|$ and $m_1 = \sup_{k \in \mathbf{R}} |\partial_k m(k)|$. Let $\mathcal{F}$ denote the (continuous) Fourier transform and $M : f \to \mathcal{F}^{-1} m(\cdot)\mathcal{F}f$. Then there exist constants C_p such that for all $1 < p < \infty$, it holds*

$$
\begin{aligned}
&\|Mf\|_\infty \leq C_\infty m_0 \big(\|f\|_2 \|\partial_y f\|_2\big)^{\frac{1}{2}}, \qquad \|Mf\|_p \leq C_p m_0 \|f\|_p \\
&\|Mf\|_1 \leq C_1 \Big(m_0 \big(\|f\|_2 \|\rho_1 f\|_2\big)^{\frac{1}{2}} + \big(m_0 m_1\big)^{\frac{1}{2}} \|f\|_2\Big).
\end{aligned}
$$

Proof. The L^p estimate for $1 < p < \infty$ is a consequence of the classical Mikhlin–Hörmander condition (see, e.g. [11]), the L^∞ and L^1 estimates are immediate consequences of Sobolev's and Plancherel's inequalities. $\square$

Lemma 3.5. *Let $\tilde{\mathcal{L}}_v = \mathcal{L}_v - \mathbb{1}$ and $\tilde{\mathcal{L}}_u = \mathcal{L}_u + \mathcal{I}\mathcal{P}_0$ and assume that* (1.16) *holds, then*

$$
\begin{aligned}
\|\mathcal{L}_u w\|_1 + \|\mathcal{L}_u w\|_{\infty,\frac{1}{2}} + \|\partial_y \mathcal{L}_u w\|_{r,1-\frac{1}{2r}-\eta} &\leq C(\|\mathcal{I}\mathcal{P}_0 w\|_1 + \|(0,0,w)\|), \\
\|\tilde{\mathcal{L}}_u w\|_1 + \|\tilde{\mathcal{L}}_u w\|_{\infty,\frac{1}{2}} + \|\partial_y \tilde{\mathcal{L}}_u w\|_{r,1-\frac{1}{2r}-\eta} &\leq C\|(0,0,w)\|, \\
\|\tilde{\mathcal{L}}_v w\|_{1,\frac{1}{2}-\varphi} + \|\tilde{\mathcal{L}}_v w\|_{\infty,1-\varphi} + \|\partial_y \tilde{\mathcal{L}}_v w\|_{r,\frac{3}{2}-\frac{1}{2r}-\xi} &\leq C\|(0,0,w)\|, \\
\|\mathcal{L}_v w\|_{1,\frac{1}{2}-\varphi} + \|\mathcal{L}_v w\|_{\infty,1-\varphi} + \|\partial_y \mathcal{L}_v w\|_{r,\frac{3}{2}-\frac{1}{2r}-\xi} &\leq C\|(0,0,w)\|.
\end{aligned}
$$

Proof. The symbol $T(k,n)$ of $\tilde{\mathcal{L}_u}$ is given by $T(k,n) = \frac{-ik}{\Lambda_- + in\tau}$ if $n \neq 0$ and $T(k,0) = \frac{-ik}{\Lambda_+}$, and it satisfies (uniformly in $n \in \mathbf{Z}$) the hypothesis of Lemma 3.4 with $m_0 = C\frac{\langle\tau\rangle}{\tau} \leq C\langle x_0\rangle^{\frac{1}{2}}$ and $m_1 = C\frac{\langle\tau\rangle^2}{\tau^2} \leq C\langle x_0\rangle$. Similarly, $\mathcal{P}\tilde{\mathcal{L}_v}$ satisfies the hypothesis of Lemma 3.4 with $m_0 = 2\frac{\langle\tau\rangle}{\tau} \leq 2\langle x_0\rangle^{\varphi} < 2\langle x_0\rangle^{\frac{1}{2}}$ and $m_1 = m_0^2 \leq 4\langle x_0\rangle^{2\varphi}$. The proof is then a straightforward application of $\partial_y \mathcal{I} f = f$, Lemma 3.2 and 3.4, inequality (3.5) and the fact that $\|w\|_{1,\frac{1}{2}} + \|w\|_{\infty,1} \leq C\|(0,0,w)\|_{x_0}$. $\square$

Lemma 3.6. *If* (1.16) *holds, then* $|||(u_{L,2}, v_{L,2}, 0)||| \leq C\|(\nu, \mathcal{H}\nu, 0)\|$ *and for all* $x \geq 2x_0$*, we have* $\|u_{L,2}(x)\|_{\infty,1-\varphi} + \|v_{L,2}(x)\|_{\infty,1-\varphi} \leq C\|(\nu,\mathcal{H}\nu,0)\|$.

Proof. We first note that $|K_0(x)|_s \leq Cx^{\frac{1}{s}-1}$ and $|\partial_y K_0(x)|_s \leq Cx^{\frac{1}{s}-2}$. Then let $q \leq p_0 \leq \infty$ and $p \leq p_1 \leq \infty$, since $(x-x_0)^{\frac{1}{s}-1} \leq C\langle x\rangle^{\frac{1}{s}-1}$ if $x \geq 2x_0$, we get

$$\begin{aligned}
|||u_{L,2}|||_{p_0,\frac{1}{2}-\frac{1}{p_0}} &\leq |||\nu|||_{p_0,\frac{1}{2}-\frac{1}{p_0}} + C\sup_{x\geq 2x_0}\langle x\rangle^{\frac{1}{2}-\frac{1}{q}}\|\nu\|_q,\\
|||v_{L,2}|||_{p_1,1-\frac{1}{p_1}-\varphi} &\leq |||\mathcal{H}\nu|||_{p_1,1-\frac{1}{p_1}-\varphi} + C\sup_{x\geq 2x_0}\langle x\rangle^{1-\frac{1}{p}-\varphi}\|\mathcal{H}\nu\|_p,\\
|||\partial_y u_{L,2}|||_{r,1-\frac{1}{2r}-\eta} &\leq |||\partial_y\nu|||_{r,1-\frac{1}{2r}-\eta} + C\sup_{x\geq 2x_0}\langle x\rangle^{\frac{1}{2}-\frac{1}{q}}\|\nu\|_q,\\
|||\partial_y v_{L,2}|||_{r,\frac{3}{2}-\frac{1}{2r}-\xi} &\leq |||\partial_y\mathcal{H}\nu|||_{r,\frac{3}{2}-\frac{1}{2r}-\xi} + C\sup_{x\geq 2x_0}\langle x\rangle^{1-\frac{1}{p}-\varphi}\|\mathcal{H}\nu\|_p,
\end{aligned}$$

while for $x \geq 2x_0$, we have

$$\|u_{L,2}(x)\|_\infty + \|v_{L,2}(x)\|_\infty \leq \langle x\rangle^{-1+\varphi}\sup_{x\geq 2x_0}\Big(\langle x\rangle^{1-\frac{1}{p}-\varphi}\big(\|\nu\|_p + \|\mathcal{H}\nu\|_p\big)\Big)$$

The proof is completed using Lemma 3.4, $\xi \geq \varphi$, $1 \leq q < 2$ and $1-\frac{1}{p} \leq \varphi < \frac{1}{2}$. $\square$

3.3. The nonlinear terms

In this section, we prove that there exist constants C and $\kappa > 0$ such that

$$|||(u_N, v_N, \omega_N)||| \leq C\langle x_0\rangle^{-\kappa}|||(u,v,\omega)|||^2, \tag{3.6}$$

This is the hardest part of the paper in that the parameters in (3.1)–(3.2) need to be chosen in the right way to get a bound that *decays* as $x_0 \to \infty$. The estimates of the 'local' terms $u_{N,6}$ and $v_{N,6}$ are given in Proposition 3.8 below. The estimates for the $\mathcal{J}_1$ terms are split component-wise in Propositions 3.9, 3.10 and 3.11, those for the $\mathcal{J}_2$ terms in Propositions 3.12, 3.13 and 3.14. During the course of these proofs, we will encounter repeatedly the following functions

$$A\begin{bmatrix}p_2,q_2,s\\p_1,q_1\end{bmatrix}(x,x_0) = \int_{x_0}^{x}\mathrm{d}\tilde{x}\ \min\Big(\frac{\langle\tilde{x}\rangle^{-q_1}}{(x-\tilde{x})^{p_1}}, \frac{\langle x\rangle^{s}\langle\tilde{x}\rangle^{-q_2}}{(x-\tilde{x})^{p_2}}\Big),$$

$$B\begin{bmatrix} p_2,q_2,s_2 \\ p_1,q_1,s_1 \end{bmatrix}(x,x_0) = \int_{x_0}^{x} \mathrm{d}\tilde{x}\; e^{\frac{b(\tau)(x-\tilde{x})}{4}} \min\Big(\frac{\langle\tilde{x}\rangle^{-q_1}\langle x-\tilde{x}\rangle^{s_1}}{(x-\tilde{x})^{p_1}}, \frac{\langle\tilde{x}\rangle^{-q_2}\langle x-\tilde{x}\rangle^{s_2}}{(x-\tilde{x})^{p_2}} \Big),$$

which occur naturally from (3.1)–(3.2). These functions satisfy the

Lemma 3.7. *Let $p_1 < 1$, $s \geq 0$ and $p_2, q_1, q_2 \in \mathbf{R}$, there exists a constant C such that for all $x \geq x_0 \geq 1$, it holds*

$$A\begin{bmatrix} p_2,q_2,s \\ p_1,q_1 \end{bmatrix}(x,x_0) \leq C\left(\langle x\rangle^{1-q_1-p_1} + \langle x\rangle^{s-p_2} \max(\langle x\rangle^{1-q_2}, \langle x_0\rangle^{1-q_2}) \right), \tag{3.7}$$

if $q_2 \neq 1$, while the same inequality holds with $\max(\langle x\rangle^{1-q_2}, \langle x_0\rangle^{1-q_2})$ *replaced by* $\ln(1+x)$ *if $q_2 = 1$. If furthermore we have $s_1, s_2 \geq 0$ and $\frac{\langle\tau\rangle}{\tau} \leq \langle x_0\rangle^{\varphi}$, then for all $m \geq 0$, there exists a constant C such that for all $x \geq x_0 \geq 1$, it holds*

$$\begin{aligned} B\begin{bmatrix} p_2,q_2,s_2 \\ p_1,q_1,s_1 \end{bmatrix}(x,x_0) \leq C\Big(& \langle x\rangle^{-q_1}\langle x_0\rangle^{2(1+s_1-p_1)\varphi} \\ & + \langle x\rangle^{-p_2-m}\langle x_0\rangle^{2(1+m+s_2)\varphi} \max(\langle x\rangle^{-q_2}, \langle x_0\rangle^{-q_2}) \Big). \end{aligned}$$

Proof. We first note that for all $p > -1$, there exists a constant C such that

$$\int_{x_0}^{x} \mathrm{d}\tilde{x}\; e^{\frac{b(\tau)(x-\tilde{x})}{4}} (x-\tilde{x})^p \leq C \int_0^\infty \mathrm{d}z\; e^{\frac{-|b(\tau)|z}{4}} z^p \leq C\langle x_0\rangle^{2(1+p)\varphi}, \tag{3.8}$$

since $|b(\tau)| \leq C\tau^{-2} \leq C\langle x_0\rangle^{2\varphi}$. We then note that since $x \geq x_0 \geq 1$, we have $\frac{\langle x\rangle}{\sqrt{2}} \leq x \leq \langle x\rangle$. We first consider the case of finite x, that is precisely, $x_0 \leq x \leq 2x_0$, then

$$\begin{aligned} A\begin{bmatrix} p_2,q_2,s \\ p_1,q_1 \end{bmatrix}(x,x_0) &\leq C\langle x_0\rangle^{-q_1}(x-x_0)^{1-p_1} \leq C\langle x_0\rangle^{1-p_1-q_1}, \\ B\begin{bmatrix} p_2,q_2,s_2 \\ p_1,q_1,s_1 \end{bmatrix}(x,x_0) &\leq \langle x_0\rangle^{-q_1} \int_{x_0}^{x} \mathrm{d}\tilde{x}\; e^{\frac{b(\tau)(x-\tilde{x})}{4}} (x-\tilde{x})^{s_1-p_1} \\ &\leq C\langle x_0\rangle^{-q_1+2(1+s_1-p_1)\varphi}. \end{aligned}$$

However, in the applications of this Lemma, we will generically have e.g. $1-q_1-p_1 < 0$, that is, the integrals we seek to bound *decay* as $x \to \infty$. To get the optimal decay rate, the idea is to consider $x \geq 2x_0$, and split the integration domain $x_0 \leq \tilde{x} \leq x$ in two equal parts. Since $x \geq 2x_0$ implies $\frac{x}{2} \leq (x-x_0) \leq x$ and $x_0 \leq \tilde{x} \leq \frac{x+x_0}{2}$ implies $\frac{x}{4} \leq \frac{x-x_0}{2} \leq x-\tilde{x} \leq x-x_0 \leq x$, we have

$$A\begin{bmatrix} p_2,q_2,s \\ p_1,q_1 \end{bmatrix}(x,x_0) \leq C\langle x\rangle^{s-p_2} \int_{x_0}^{\frac{x+x_0}{2}} \mathrm{d}\tilde{x}\; \langle\tilde{x}\rangle^{-q_2} + C\langle x\rangle^{-q_1} \int_{\frac{x+x_0}{2}}^{x} \mathrm{d}\tilde{x}\; (x-\tilde{x})^{-p_1}.$$

The proof of (3.7) is completed using $\int_{x_0}^{\frac{x+x_0}{2}} \mathrm{d}\tilde{x}\; \langle\tilde{x}\rangle^{-q_2} \leq \int_{x_0}^{x} \mathrm{d}\tilde{x}\; \langle\tilde{x}\rangle^{-q_2}$ and considering separately $q_2 < 1$, $q_2 = 1$ and $q_2 > 1$. In the same way, we have

$$B\begin{bmatrix} p_2,q_2,s_2 \\ p_1,q_1,s_1 \end{bmatrix}(x,x_0) \leq \frac{C\max(\langle x\rangle^{-q_2}, \langle x_0\rangle^{-q_2})}{\langle x\rangle^{p_2+m}} \int_{x_0}^{\frac{x+x_0}{2}} \mathrm{d}\tilde{x}\; e^{\frac{b(\tau)(x-\tilde{x})}{4}} (x-\tilde{x})^{s_2+m}$$

$$+\frac{C}{\langle x\rangle^{q_1}}\int_{\frac{x+x_0}{2}}^{x}\mathrm{d}\tilde{x}\; e^{\frac{b(\tau)(x-\tilde{x})}{4}}(x-\tilde{x})^{s_1-p_1},$$

which completes the proof with the help of (3.8). □

From now on, we begin the estimates on $u_{N,i}$, $v_{N,i}$ and $\omega_{N,i}$. We first have the

Proposition 3.8. *Assume that* (1.16) *holds, then for* $\kappa_0=\min(\frac{\varphi}{2},\frac{1}{2}-\eta+\xi-\varphi)$, *we have* $\|(u_{N,6},v_{N,6},0)\|\le C\langle x_0\rangle^{-\kappa_0}\|(u,v,\omega)\|^2$ *and* $\|u_{N,6}\|_{\infty,1}+\|v_{N,6}\|_{\infty,1}\le C\|(u,v,\omega)\|^2$.

Proof. The proof follows at once from Lemma A.3 and (3.4). □

Note that $u_{N,6}$ and $v_{N,6}$ decay faster than u and v as $x\to\infty$. We then turn to the estimates of $\omega_{N,1}+\omega_{N,2}+\omega_{N,3}$. To prepare the ground for the asymptotic results of Section 4, we also show that $\omega_{N,2}$ and $\omega_{N,3}$ decay faster than ω as $x\to\infty$.

Proposition 3.9. *If* (1.16) *holds, then there exists a constant* C *such that for* $\kappa_{1,1}=\min(\frac{1}{4}-\frac{\varphi}{2}-\eta,\frac{1}{2}-\xi)$, *we have*

$$\begin{aligned}
\|(0,0,\omega_{N,1}+\omega_{N,2}+\omega_{N,3})\|&\le C\langle x_0\rangle^{-\kappa_{1,1}}\|(u,v,\omega)\|^2,\\
\|\omega_{N,2}\|_{\infty,\frac{3}{2}-\varphi}+\|\omega_{N,2}\|_{1,1-\varphi}+\|\rho_\beta\omega_{N,2}\|_{2,\frac{5}{4}-\frac{\beta}{2}-\varphi}&\le C\|(u,v,\omega)\|^2,\\
\|\omega_{N,3}\|_{\infty,\frac{3}{2}-\varphi}+\|\omega_{N,3}\|_{1,1-\varphi}+\|\rho_\beta\omega_{N,3}\|_{2,\frac{5}{4}-\frac{\beta}{2}-\varphi}&\le C\|(u,v,\omega)\|^2.
\end{aligned}$$

Proof. We give the proof only for the case $\|(u,v,\omega)\|=1$, from which the general case follows immediately. From Appendix A and (3.1)–(3.2), it follows easily that

$$\begin{aligned}
\|(0,0,\omega_{N,1}(x))\|\le{}& C\Big(\langle x\rangle^{\frac{3}{4}}A\begin{bmatrix}\frac{3}{4},\frac{3}{2}-\varphi,0\\ \frac{3}{4},\frac{3}{2}-\varphi\end{bmatrix}(x,x_0)+\langle x\rangle^{\frac{3}{4}-\frac{\beta}{2}}A\begin{bmatrix}\frac{3}{4}-\frac{\beta}{2},\frac{3}{2}-\varphi,0\\ \frac{3}{4}-\frac{\beta}{2},\frac{3}{2}-\varphi\end{bmatrix}(x,x_0)\Big)\\
&+C\Big(\langle x\rangle^{\frac{3}{4}-\frac{\beta}{2}}A\begin{bmatrix}\frac{1}{2},\frac{7}{4}-\frac{\beta}{2}-\varphi,0\\ \frac{1}{2},\frac{7}{4}-\frac{\beta}{2}-\varphi\end{bmatrix}(x,x_0)\Big)\\
&+C\Big(\langle x\rangle^{\frac{3}{2}}A\begin{bmatrix}2,\frac{3}{2}-\varphi,\frac{1}{2}\\ \frac{3}{4},\frac{9}{4}-\xi\end{bmatrix}(x,x_0)+\langle x\rangle A\begin{bmatrix}\frac{3}{2},\frac{3}{2}-\varphi,\frac{1}{2}\\ \frac{1}{2},2-\xi\end{bmatrix}(x,x_0)\Big).
\end{aligned}$$

Using Lemma 3.7 and $\beta\ge\frac{3}{2}$, we get $\|(0,0,\omega_{N,1})\|\le C\Big(\langle x_0\rangle^{-\frac{1}{2}+\varphi}+\langle x_0\rangle^{-\frac{1}{2}+\xi}\Big)$. Similarly, from Lemma A.6, it follows easily, choosing $\xi_2=1-\varepsilon_1$ and $\xi_3=2-2\varepsilon_2$ with $\varepsilon_i>0$, that

$$\begin{aligned}
\|(0,0,\omega_{N,2}(x))\|\le{}& C\Big(\langle x\rangle^{\frac{3}{4}}A\begin{bmatrix}1-\varepsilon_1,\frac{5}{4},0\\ 1-\varepsilon_1,\frac{5}{4}\end{bmatrix}(x,x_0)+\langle x\rangle^{\frac{3}{4}-\frac{\beta}{2}}A\begin{bmatrix}\frac{5}{4}-\frac{\beta}{2},1,0\\ \frac{5}{4}-\frac{\beta}{2},1\end{bmatrix}(x,x_0)\Big)\\
&+C\Big(\langle x\rangle^{\frac{3}{4}-\frac{\beta}{2}}A\begin{bmatrix}1-\varepsilon_1,\frac{5}{4}-\frac{\beta}{2},0\\ 1-\varepsilon_1,\frac{5}{4}-\frac{\beta}{2}\end{bmatrix}(x,x_0)\Big)\\
&+C\Big(\langle x\rangle^{\frac{3}{2}}A\begin{bmatrix}2,\frac{3}{2},\frac{1}{2}\\ 1-\varepsilon_2,\frac{7}{4}-\eta\end{bmatrix}(x,x_0)+\langle x\rangle A\begin{bmatrix}2,1,\frac{1}{2}\\ 1-\varepsilon_1,\frac{3}{2}-\eta\end{bmatrix}(x,x_0)\Big),
\end{aligned}$$

$$\|\omega_{N,2}(x)\|_\infty \le CA\begin{bmatrix}2,1,\frac{1}{2}\\1-\varepsilon_1,\frac{3}{2}\end{bmatrix}(x,x_0) \le C\big(\langle x\rangle^{-\frac{3}{2}+\varepsilon_1}+\langle x\rangle^{-\frac{3}{2}}\ln(x)\big),$$

$$\|\omega_{N,2}(x)\|_1 \le CA\begin{bmatrix}1,1,0\\1-\varepsilon_1,1\end{bmatrix}(x,x_0) \le C\big(\langle x\rangle^{-1+\varepsilon_1}+\langle x\rangle^{-1}\ln(x)\big),$$

$$\|\rho_\beta\omega_{N,2}(x)\|_2 \le C\Big(A\begin{bmatrix}\frac{5}{4}-\frac{\beta}{2},1,0\\\frac{5}{4}-\frac{\beta}{2},1\end{bmatrix}(x,x_0)+A\begin{bmatrix}1-\varepsilon_1,\frac{5}{4}-\frac{\beta}{2},0\\1-\varepsilon_1,\frac{5}{4}-\frac{\beta}{2}\end{bmatrix}(x,x_0)\Big).$$

Let $\tilde\kappa(\varepsilon_1,\varepsilon_2)=\min(\frac{1}{2}-\varphi,\frac{1}{2}-\varepsilon_1-\eta,\frac{1}{4}-\varepsilon_2-\eta)$. By Lemma 3.7, $\ln(1+x)\le C\langle x\rangle^\varphi$ and $\varepsilon_i>0$, we get

$$\|\!|(0,0,\omega_{N,2})\|\!| \le C\langle x_0\rangle^{-\tilde\kappa(\varepsilon_1,\varepsilon_2)}, \tag{3.9}$$

$$\|\!|\omega_{N,2}\|\!|_{\infty,\frac{3}{2}-\varepsilon_1}+\|\!|\omega_{N,2}\|\!|_{1,1-\varepsilon_1}+\|\!|\rho_\beta\omega_{N,2}\|\!|_{2,\frac{5}{4}-\frac{\beta}{2}-\varepsilon_1}\le C. \tag{3.10}$$

Finally, from Lemma A.7, it follows that

$$\begin{aligned}\|(0,0,\omega_{N,3}(x))\| \le{}& C\Big(\langle x\rangle^{\frac{3}{4}}B\begin{bmatrix}\frac{3}{4},1,0\\\frac{3}{4},1,0\end{bmatrix}(x,x_0)+\langle x\rangle^{\frac{3}{4}-\frac{\beta}{2}}B\begin{bmatrix}\frac{9}{8}-\frac{3\beta}{8},1,\frac{3}{8}+\frac{\beta}{8}\\\frac{9}{8}-\frac{3\beta}{8},1,\frac{3}{8}+\frac{\beta}{8}\end{bmatrix}(x,x_0)\Big)\\&+C\Big(\langle x\rangle^{\frac{3}{4}-\frac{\beta}{2}}B\begin{bmatrix}\frac{5}{8},\frac{5}{4}-\frac{\beta}{2},\frac{1}{8}\\\frac{5}{8},\frac{5}{4}-\frac{\beta}{2},\frac{1}{8}\end{bmatrix}(x,x_0)\Big)\\&+C\Big(\langle x\rangle^{\frac{3}{2}}B\begin{bmatrix}2,1,\frac{1}{2}\\\frac{3}{4},\frac{7}{4}-\eta,0\end{bmatrix}(x,x_0)+\langle x\rangle B\begin{bmatrix}\frac{13}{8},1,\frac{5}{8}\\\frac{5}{8},\frac{3}{2}-\eta,\frac{1}{8}\end{bmatrix}(x,x_0)\Big),\end{aligned}$$

$$\|\omega_{N,3}(x)\|_\infty \le CB\begin{bmatrix}1,1,0\\\frac{5}{8},\frac{3}{2},\frac{1}{8}\end{bmatrix}(x,x_0)\le C\langle x\rangle^{-\frac{3}{2}+\varphi},$$

$$\|\omega_{N,3}(x)\|_1 \le CB\begin{bmatrix}\frac{5}{8},1,\frac{1}{8}\\\frac{5}{8},1,\frac{1}{8}\end{bmatrix}(x,x_0)\le C\langle x\rangle^{-1+\varphi},$$

$$\begin{aligned}\|\rho_\beta\omega_{N,3}(x)\|_2 &\le C\Big(B\begin{bmatrix}\frac{9}{8}-\frac{3\beta}{8},1,\frac{3}{8}+\frac{\beta}{8}\\\frac{9}{8}-\frac{3\beta}{8},1,\frac{3}{8}+\frac{\beta}{8}\end{bmatrix}(x,x_0)+B\begin{bmatrix}\frac{5}{8},\frac{5}{4}-\frac{\beta}{2},\frac{1}{8}\\\frac{5}{8},\frac{5}{4}-\frac{\beta}{2},\frac{1}{8}\end{bmatrix}(x,x_0)\Big),\\&\le C\langle x\rangle^{-\frac{5}{4}+\frac{\beta}{2}+\varphi},\end{aligned}\tag{3.11}$$

where in (3.11), we used $\beta\ge\frac{1}{2}$. Using Lemma 3.7 and $\beta\ge1$, we get $\|\!|(0,0,\omega_{N,3})\|\!|\le C\langle x_0\rangle^{-\frac{1}{4}+\frac{\varphi}{2}+\eta}$. Choosing $\varepsilon_1=\varphi$ and $\varepsilon_2=\frac{\varphi}{2}$ in (3.9)–(3.10) completes the proof. □

We now turn to $v_{N,1}+v_{N,2}+v_{N,3}+v_{N,7}$. For further reference, we also show that some of these terms have improved decay rates compared to those of v.

Proposition 3.10. *If* (1.16) *holds, then there exists a constant C such that for $\kappa_{1,2}=\min(\kappa_{1,1},\frac{\varphi}{2},\frac{1}{2}-\eta+\xi-2\varphi)$, we have*

$$\|\!|(0,v_{N,1}+v_{N,2}+v_{N,3}+v_{N,7},0)\|\!| \le C\langle x_0\rangle^{-\kappa_{1,2}}\|\!|(u,v,\omega)\|\!|^2, \tag{3.12}$$

$$\|\!|v_{N,1}+v_{N,3}\|\!|_{\infty,\frac{3}{2}-\varphi}\le C\langle x_0\rangle^{\varphi}\|\!|(u,v,\omega)\|\!|^2.$$

Proof. We give the proof only for the case $\|\!|(u,v,\omega)\|\!|=1$. Since $v_{N,7}(x)=\omega_{N,1}(x)+\omega_{N,2}(x)+\omega_{N,3}(x)$, we see using (3.3) that the contribution of $v_{N,7}$

to (3.12) is already proved in Proposition 3.9. Then, from Lemmas A.8 and A.9, it follows easily that

$$|K_{10}(x)+K_{11}(x)|_1 \le Ce^{\frac{b(\tau)(x-\tilde{x})}{4}}\left(\frac{1}{x^{\frac{1}{2}}}+\frac{\langle x\rangle^{\frac{1}{8}}}{x^{\frac{1}{8}}}+\frac{\langle x\rangle^{\frac{1}{8}}\langle x_0\rangle^{\varphi}}{x^{\frac{1}{4}}}\right) \equiv CB_1(x),$$
$$|K_9(x)|_1 \le Ce^{\frac{b(\tau)(x-\tilde{x})}{4}}\left(1+\frac{\langle x_0\rangle^{\varphi}}{x^{\frac{1}{4}}}\right) \equiv CD_1(x).$$

Let $\delta_s(x) = \langle x\rangle^{-s}$, $\mathcal{B}_s(x) = \mathcal{J}_1[B_1,\delta_s](x)$ and $\mathcal{D}_s(x) = \mathcal{J}_1[D_1,\delta_s](x)$. We have $\|v_{N,3}(x)\|_\infty \le C\mathcal{B}_{\frac{3}{2}}(x)$, $\|v_{N,1}(x)\|_\infty \le C\mathcal{D}_{2-\varphi}(x)$ and

$$\begin{aligned}\|(0,v_{N,3}(x),0)\| &\le C\langle x\rangle^{1-\varphi}\mathcal{B}_{\frac{3}{2}}(x)+C\langle x\rangle^{1-\varphi-\frac{1}{p}}\mathcal{B}_{\frac{3}{2}-\frac{1}{2p}}(x)\\ &\quad + C\langle x\rangle^{\frac{3}{2}-\frac{1}{2r}-\xi}\mathcal{B}_{2-\frac{1}{2r}-\eta}(x),\\ \|(0,v_{N,1}(x),0)\| &\le C\langle x\rangle^{1-\varphi}\mathcal{D}_{2-\varphi}(x)+C\langle x\rangle^{1-\varphi-\frac{1}{p}}\mathcal{D}_{2-\varphi-\frac{1}{2p}}(x)\\ &\quad + C\langle x\rangle^{\frac{3}{2}-\frac{1}{2r}-\xi}\mathcal{D}_{\frac{5}{2}-\frac{1}{2r}-\xi}(x).\end{aligned}$$

Let $\tilde{\kappa}_{1,2} = \min(\frac{1}{2}-\varphi, \frac{1}{2}-\eta+\xi-2\varphi)$. Using Lemma 3.7, we get $\|v_{N,3}\|_{\infty,\frac{3}{2}-\varphi} + \|v_{N,1}\|_{\infty,\frac{3}{2}-\varphi} \le C\langle x_0\rangle^{\varphi}$ and $\|(0,v_{N,3},0)\| + \|(0,v_{N,1},0)\| \le C\langle x_0\rangle^{-\tilde{\kappa}_{1,2}}$. In the same way, from Lemma A.4, it follows easily that for all $q>1$ and $s\ge 1$, we have

$$|\partial_y K_{13}(x)|_2 \le Cx^{-\frac{3}{2}}, \qquad |K_{13}(x)|_q \le Cx^{-1+\frac{1}{q}}\left(1+\frac{\langle x_0\rangle^{\frac{1}{4q}}}{x^{\frac{1}{4q}}}\right) \equiv CE_q(x).$$

Then, let $\mathcal{E}_{q,s}(x) = \mathcal{J}_1[E_q,\rho_s](x)$. We have $\|v_{N,2}(x)\|_{\infty,1-\varphi} \le C\langle x\rangle^{1-\varphi}\mathcal{E}_{\frac{1}{\varphi},\frac{3}{2}-\frac{\varphi}{2}}(x)$, $\|v_{N,2}(x)\|_{p,1-\varphi-\frac{1}{p}} \le C\langle x\rangle^{1-\varphi-\frac{1}{p}}\mathcal{E}_{p,\frac{3}{2}-\varphi}(x)$. Using $\chi = \frac{3}{2}-\frac{1}{2r}-\xi$

$$\|\partial_y v_{N,2}(x)\|_{r,\frac{3}{2}-\frac{1}{2r}-\xi} \le C\langle x\rangle^{\frac{3}{2}-\frac{1}{2r}-\xi}\int_{x_0}^{x}\mathrm{d}\tilde{x}\ \min\left(\frac{E_2(x-\tilde{x})}{\langle\tilde{x}\rangle^{\frac{9}{4}-\frac{1}{2r}-\xi}}, \frac{\tilde{x}^{-\frac{7}{4}+\frac{1}{2r}}}{(x-\tilde{x})^{\frac{3}{2}}}\right).$$

Using Lemma 3.7 and $r>2$, we get $\|(0,v_{N,2}(x),0)\| \le C\langle x_0\rangle^{-\frac{\varphi}{2}}$. □

We conclude this section by estimating $u_{N,1}+u_{N,2}+u_{N,3}$. In the spirit of Proposition 3.9, we will also show that some of these terms have improved decay rates as $x\to\infty$.

Proposition 3.11. *Assume that the parameters satisfy* (1.16)*, then there exists a constant C such that for $\kappa_{1,3} = \min(\kappa_{1,2}, \frac{1}{2}-(1+\frac{1}{2r})\varphi, \frac{1}{2}-\xi+\eta-\frac{\varphi}{r})$ and all $\varepsilon>0$, we have*

$$\|(u_{N,1}+u_{N,2}+u_{N,3},0,0)\| \le \langle x_0\rangle^{-\kappa_{1,3}}\|(u,v,\omega)\|^2, \tag{3.13}$$
$$\|u_{N,2}+u_{N,3}\|_{\infty,1-\varphi} + \|\mathcal{P}(u_{N,1}+u_{N,3})\|_{\infty,\frac{3}{2}} \le C\langle x_0\rangle^{\frac{5\varphi}{2}}\|(u,v,\omega)\|^2.$$

Proof. We also give the proof only for the case $\|(u,v,\omega)\|=1$. We first note that $\|(u_{N,2},0,0)\| \le \|(0,u_{N,2},0)\|$ (see (3.3)), and that $u_{N,2}$ and $v_{N,2}$ differ only by signs and the exchange of the Kernels K_{12} and K_{13}. The bound on $v_{N,2}$ in the proof of Proposition 3.10 being insensitive to these details then applies mutatis mutandis, in particular, we have $\|u_{N,2}\|_{\infty,1-\varphi} \le C$. Then, by Lemma A.8, we have

$$|K_8(x)|_p \le C\frac{\langle x\rangle^{\frac12-\frac{1}{2p}}}{x^{1-\frac1p}}\Big(1+\frac{\langle x_0\rangle^{\frac{\varphi}{p}}}{x^{\frac{1}{4p}}}\Big) \equiv CE_p(x),$$

$$|K_2(x)|_p + |K_9(x)|_p \le C\left(\frac{1}{x^{1-\frac{1}{2p}}} + \frac{e^{\frac{b(\tau)x}{4}}\langle x\rangle^{\frac12-\frac{1}{2p}}}{x^{1-\frac1p}}\Big(1+\frac{\langle x_0\rangle^{\frac{\varphi}{p}}}{x^{\frac{1}{4p}}}\Big)\right) \equiv CH_p(x),$$

$$|\partial_y K_2(x)|_2 + |\partial_y K_9(x)|_2 \le C\left(\frac{\langle x\rangle^{\frac12}}{x^{\frac74}} + \frac{e^{\frac{b(\tau)x}{4}}\langle x\rangle^{\frac34}}{x^{\frac32}}\right) \equiv CJ(x),$$

so that

$$\|u_{N,1}(x)\|_{p_0,\frac12-\frac{1}{p_0}} \le C\langle x\rangle^{\frac12-\frac{1}{p_0}}\int_{x_0}^{x}\mathrm{d}\tilde x\,\min\Big(\frac{E_1(x-\tilde x)}{\langle\tilde x\rangle^{2-\varphi-\frac{1}{2p_0}}},\frac{E_{p_0}(x-\tilde x)}{\langle\tilde x\rangle^{\frac32-\varphi}}\Big)$$

$$\|\partial_y u_{N,1}(x)\|_{r,1-\frac{1}{2r}-\eta} \le C\langle x\rangle^{1-\frac{1}{2r}-\eta}\int_{x_0}^{x}\mathrm{d}\tilde x\;\min\Big(\frac{E_r(x-\tilde x)}{\langle\tilde x\rangle^{2-\xi}},\frac{\langle x\rangle^{1-\frac{1}{2r}}\langle\tilde x\rangle^{-\frac32+\varphi}}{(x-\tilde x)^{2-\frac1r}}\Big),$$

$$\|u_{N,3}(x)\|_{p_0} \le C\int_{x_0}^{x}\mathrm{d}\tilde x\;\min\Big(\langle\tilde x\rangle^{-\frac32+\frac{1}{2p_0}}H_1(x-\tilde x),\langle\tilde x\rangle^{-1}H_{p_0}(x-\tilde x)\Big)$$

$$\|\partial_y u_{N,3}(x)\|_{r,1-\frac{1}{2r}-\eta} \le C\langle x\rangle^{1-\frac{1}{2r}-\eta}\int_{x_0}^{x}\mathrm{d}\tilde x\;\min\Big(\frac{H_r(x-\tilde x)}{\langle\tilde x\rangle^{\frac32-\eta}},\frac{J(x-\tilde x)}{\langle\tilde x\rangle^{\frac54-\frac{1}{2r}}}\Big).$$

By Lemma 3.7, using these bounds with $p_0 = q$ and $p_0 = \infty$, we get

$$\|(u_{N,1},0,0)\| \le C\Big(\langle x_0\rangle^{-\frac12+\varphi} + \langle x_0\rangle^{-\frac12+\xi-\eta+\frac{\varphi}{r}-\frac{1}{4r}}\Big),$$

$$\|(u_{N,3},0,0)\| \le C\Big(\langle x_0\rangle^{-\frac12+\varphi} + \langle x_0\rangle^{-\frac12-\frac{1}{2r}+(1+\frac{3}{2r})\varphi}\Big),$$

and $\|u_{N,3}\|_{\infty,1-\varphi} \le C$. We finally note that

$$\|\mathcal{P}(u_{N,1}(x)+u_{N,3}(x))\|_\infty \le C\int_{x_0}^{x}\mathrm{d}\tilde x\;e^{\frac{b(\tau)(x-\tilde x}{4}}\Big(1+\frac{1}{x^{\frac12}}+\frac{\langle x_0\rangle^{\varphi}}{(x-\tilde x)^{\frac14}}\Big)\langle\tilde x\rangle^{-\frac32},$$

which shows that $\|\mathcal{P}(u_{N,1}(x)+u_{N,3}(x))\|_{\infty,\frac32} \le C\langle x_0\rangle^{\frac{5\varphi}{2}}$ and completes the proof. □

We now turn to the estimates of $\mathcal{J}_2$ terms. For further reference, we will also point out that most decay rates are in fact better than those of the related fields. We begin with the vorticity component:

Proposition 3.12. *If* (1.16) *holds, then there exists a constant C such that for $\kappa_{2,1} = \frac14 - \eta$, we have $\|(0,0,\omega_{N,4})\| \le C\langle x_0\rangle^{-\kappa_{2,1}}\|(u,v,\omega)\|^2$ and*

$$\|\omega_{N,4}\|_{\infty,\frac32} + \|\omega_{N,4}\|_{1,1} + \|\rho_\beta\omega_{N,4}\|_{2,\frac54-\frac{\beta}{2}} \le C\|(u,v,\omega)\|^2.$$

Proof. We give the proof only for the case $\|(u,v,\omega)\| = 1$. From the results of Appendix A, it follows easily that there are exponents $p \geq 0$ and $q < 1$ such that $\|K_1+K_6+K_7\|_{1,\{p,q\}}$, $\|K_2\|_{1,\{p,q\}}$, $\|K_1+K_6+K_7\|_{2,\{p,q\}}$, $\|K_2\|_{2,\{p,q\}}$, $\|\rho_\beta(K_1+K_6+K_7)\|_{1,\{p,q\}}$ and $\|\rho_\beta K_2\|_{1,\{p,q\}}$ are all bounded by a constant. We then note that, e.g.

$$\|\mathcal{J}_2[\mathcal{E}K_2,Q]\|_{2,\frac{3}{4}} \leq C \sup_{x\geq x_0} \int_x^\infty \mathrm{d}\tilde{x}\ \frac{e^{x-\tilde{x}}\langle \tilde{x}-x\rangle^p}{(\tilde{x}-x)^q} \langle x\rangle^{\frac{3}{4}} \|Q(\tilde{x})\|_2 \leq C\langle x_0\rangle^{\varphi-1},$$

which follows from

$$\sup_{\tilde{x}\geq x} \langle x\rangle^{\frac{3}{4}} \|Q(\tilde{x})\|_2 \leq \|Q\|_{2,\frac{7}{4}-\varphi} \sup_{\tilde{x}\geq x} \langle x\rangle^{\frac{3}{4}} \langle \tilde{x}\rangle^{-\frac{7}{4}+\varphi} \leq \langle x\rangle^{\varphi-1}. \tag{3.14}$$

Using similar estimates, $\beta > \frac{3}{2}$, $\varphi \leq \xi < \frac{1}{2}$ and $\frac{1}{2} - \xi + \eta \geq 0$, we easily get

$$\begin{aligned} \|(0,0,\omega_{N,4})\| &\leq C\langle x_0\rangle^{\eta-\frac{1}{4}}, \\ \|\omega_{N,4}\|_{\infty,\frac{3}{2}} + \|\omega_{N,4}\|_{1,1} + \|\rho_\beta \omega_{N,4}\|_{2,\frac{5}{4}-\frac{\beta}{2}} &\leq C. \end{aligned} \qquad \square$$

Proposition 3.13. *Let $\kappa_{2,2} = \min(\kappa_{2,1}, \frac{\varphi}{2})$. If (1.16) holds, then there exists a constant C such that for all $0 < \varepsilon \leq 1$, we have*

$$\begin{aligned} \|(0, v_{N,4}+v_{N,5}+v_{N,8}, 0)\| &\leq C\langle x_0\rangle^{-\kappa_{2,2}} \|(u,v,\omega)\|^2, \\ \|v_{N,4}+v_{N,5}+v_{N,8}\|_{\infty,\frac{3}{2}-(1+\varepsilon)\varphi} &\leq C\|(u,v,\omega)\|^2. \end{aligned} \tag{3.15}$$

Proof. We give the proof only for the case $\|(u,v,\omega)\| = 1$. Since $v_{N,8}(x) = \omega_{N,4}(x)$, we see again that using (3.3), the contribution of $\omega_{N,4}$ to (3.15) is already proved in Proposition 3.12. We then proceed as in Proposition 3.12. There are exponents $p \geq 0$, $q < 1$ and $s > 1$ such that $\|K_4\|_{1,\{p,q\}} + \|K_5\|_{1,\{p,q\}} \leq C$ and for $x \leq \tilde{x}$,

$$\|\mathcal{P}_0 K_{13}\|_{s,\{0,1-\frac{1}{s}\}} \leq C, \qquad \|\mathcal{P}K_{13}\|_{s,\{0,1-\frac{3}{4s}\}} \leq C\langle x_0\rangle^{\frac{1}{4s}} \leq C\langle \tilde{x}\rangle^{\frac{1}{4s}},$$

where we used $\tau^{-\frac{1}{4s}} \leq \langle\tau\rangle^{-\frac{1}{4s}} \langle x_0\rangle^{\frac{\varphi}{4s}} \leq \langle x_0\rangle^{\frac{1}{4s}}$ in the last inequality. Then, for all $x \geq x_0$, we have (as in (3.14) above)

$$\sup_{\tilde{x}\geq x} \Big(\langle x\rangle^{1-\varphi} \|P(\tilde{x})\|_\infty + \langle x\rangle^{1-\varphi-\frac{1}{p}} \|P(\tilde{x})\|_p + \langle x\rangle^{\frac{3}{2}-\frac{1}{2r}-\xi} \|\partial_y P(\tilde{x})\|_r \Big) \leq C\langle x\rangle^{-\frac{1}{2}},$$
$$\sup_{\tilde{x}\geq x} \Big(\langle x\rangle^{1-\varphi} \|Q(\tilde{x})\|_\infty + \langle x\rangle^{1-\varphi-\frac{1}{p}} \|Q(\tilde{x})\|_p + \langle x\rangle^{\frac{3}{2}-\frac{1}{2r}-\xi} \|\partial_y Q(\tilde{x})\|_r \Big) \leq C\langle x\rangle^{-\frac{1}{2}},$$

since $\varphi \leq \xi < \frac{1}{2}$ and $\eta \leq \xi$. As in Proposition 3.12, we thus get $\|v_{N,5}\|_{\infty,\frac{3}{2}-\varphi} \leq C$ and for all $0 < \varepsilon \leq 1$,

$$\|(0, v_{N,5}, 0)\| \leq C\langle x_0\rangle^{-\frac{1}{2}} \sup_{x\geq x_0} \int_x^\infty \mathrm{d}\tilde{x}\ \frac{e^{x-\tilde{x}}\langle\tilde{x}-x\rangle^p}{(\tilde{x}-x)^q},$$
$$\|v_{N,4}(x)\|_{\infty,1-\varphi} \leq C\langle x\rangle^{-\frac{1}{2}+\varepsilon\varphi} \int_1^\infty \mathrm{d}z\ \frac{z^{-\frac{3}{2}+(1-\varepsilon)\varphi}}{(z-1)^{1-2\varepsilon\varphi}} \big(1 + (z-1)^{-\frac{\varphi}{4}}\big),$$

$$\|v_{N,4}(x)\|_{p,1-\varphi-\frac{1}{p}} \leq C\langle x\rangle^{-\frac{1}{2}} \int_1^\infty \mathrm{d}z\, \frac{z^{-\frac{3}{2}+\varphi}}{(z-1)^{1-\frac{1}{p}}}\left(1+(z-1)^{-\frac{1}{4p}}\right),$$
$$\|\partial_y v_{N,4}(x)\|_{r,\frac{3}{2}-\frac{1}{2r}-\xi} \leq C\langle x\rangle^{-\frac{1}{4}} \int_1^\infty \mathrm{d}z\, z^{-\frac{9}{4}+\frac{1}{2r}+\eta}\left((z-1)^{-\frac{1}{2}}+(z-1)^{-\frac{5}{8}}\right),$$

where we used the change of variables $\tilde{x} = xz$ in the three last inequalities. □

Proposition 3.14. *Let $\kappa_{2,3} = \kappa_{2,2}$. If (1.16) holds, then there exists a constant C such that for all $0 < \varepsilon \leq 1$, it holds $\|\!|(u_{N,4}+u_{N,5},0,0)\|\!| \leq C\langle x_0\rangle^{-\kappa_{2,3}}\|\!|(u,v,\omega)\|\!|^2$ and*

$$\|\!|u_{N,4}+u_{N,5}\|\!|_{\infty,\frac{3}{2}-(1+\varepsilon)\varphi} \leq C\|\!|(u,v,\omega)\|\!|^2.$$

Proof. We give the proof only for the case $\|\!|(u,v,\omega)\|\!| = 1$. We first note that $\|K_2\|_{1,\{0,\frac{1}{2}\}} + \|K_3\|_{1,\{0,\frac{1}{2}\}} + \|K_4\|_{1,\{0,\frac{1}{2}\}} \leq C$, while for all $x_0 \leq x \leq \tilde{x}$, using (3.4), we have

$$\sup_{\tilde{x}\geq x}\left(\langle x\rangle^{\frac{1}{2}}\|P(\tilde{x})\|_\infty + \langle x\rangle^{\frac{1}{2}-\frac{1}{p}}\|P(\tilde{x})\|_p + \langle x\rangle^{1-\frac{1}{2r}-\eta}\|\partial_y P(\tilde{x})\|_r\right) \leq C\langle x\rangle^{-1}$$
$$\sup_{\tilde{x}\geq x}\left(\langle x\rangle^{\frac{1}{2}}\|Q(\tilde{x})\|_\infty + \langle x\rangle^{\frac{1}{2}-\frac{1}{p}}\|Q(\tilde{x})\|_p + \langle x\rangle^{1-\frac{1}{2r}-\eta}\|\partial_y Q(\tilde{x})\|_r\right) \leq C\langle x\rangle^{-1},$$

since $\varphi \leq \xi < \frac{1}{2}$ and $\frac{1}{2}-\xi+\eta \geq 0$. Proceeding as in the proof Proposition 3.13, we easily get $\|\!|(u_{N,5},0,0)\|\!| \leq C\langle x_0\rangle^{-\kappa_{2,3}}$ and $\|\!|u_{N,5}\|\!|_{\infty,\frac{3}{2}-(1+\varepsilon)\varphi} \leq C$. Next, we use (3.3) and note that $u_{N,4}$ and $v_{N,4}$ differ only by signs and the exchange of the Kernels K_{12} and K_{13} (see (2.3) and (2.2)). The bounds on $v_{N,4}$ in the proof of Proposition 3.13 being insensitive to these details then apply mutatis mutandis. □

4. Asymptotics

We now turn to the asymptotic description of the (locally) unique solutions of (1.1) in $\mathcal{C}_u$. As explained in Subsection 1.5, we will first prove the partial description of Corollary 1.5, before turning to the full description in Section 5. To avoid the unnecessary proliferation of symbols, in this section, the letter C stands for a constant which may change its value from instance to instance and depends on x_0, $\|(\nu,\mathcal{H}\nu,w)\|$ and $\|\!|(u,v,\omega)\|\!|$. We will prove (1.17) only for $x \geq 2x_0$, as the estimates are trivially satisfied otherwise.

The proof of Corollary 1.5 is given in the next subsection. Its basis is that the large time asymptotics of $K_1(x)f$ is captured by[5] $\mathcal{M}_0(f)K_1(x)$ if f decays sufficiently fast, which is the content of the next Lemma.

[5] By abuse of notation, K_1 is here considered as a function and not as a convolution operator.

Lemma 4.1. *Let $0 \le \gamma \le 1$, $0 \le \gamma_2 \le 2$. Let*

$$\begin{aligned}\big(\mathcal{R}_1 f\big)(x) &= K_1(x)(f - \mathcal{M}_0(f)) - \partial_y K_1(x)\mathcal{M}_1(f),\\ \big(\mathcal{R}_2 f\big)(x) &= K_1(x)(f - \mathcal{M}_0(f))\\ \big(\mathcal{R}_3 f\big)(x) &= K_8(x)(f - \mathcal{M}_0(f)) - \partial_y K_8(x)\mathcal{M}_1(f).\end{aligned}$$

Then for all $m \ge 0$ and $2 \le s \le \infty$, there exist constants C_γ, C_{γ_2} such that

$$\begin{aligned}\|\partial_y^m \big(\mathcal{R}_2 f\big)(x)\|_s &\le C_\gamma \frac{\langle x\rangle^{\frac{1}{2}-\frac{1}{2s}+\frac{m+\gamma}{2}}}{x^{1-\frac{1}{s}+m+\gamma}}\ \|\rho_\gamma f\|_1,\\ \|\partial_y^m \big(\mathcal{R}_2 f\big)(x)\|_1 &\le C_\gamma \frac{\langle x\rangle^{\frac{3+\gamma}{4}+\frac{m}{2}}}{x^{1+m+\frac{\gamma}{2}}}\ (\|\rho_1 f\|_1\|\rho_\gamma f\|_1)^{\frac{1}{2}},\\ \|\rho_1\partial_y^m \big(\mathcal{R}_2 f\big)(x)\|_2 &\le C_\gamma \frac{\langle x\rangle^{\frac{5}{4}+\frac{m}{2}}}{x^{\frac{3}{2}+m}}\ \|\rho_1 f\|_1,\\ \|\big(\mathcal{R}_1 f\big)(x)\|_\infty + \|\big(\mathcal{R}_3 f\big)(x)\|_\infty &\le C_{\gamma_2} \frac{\langle x\rangle^{\frac{1+\gamma_2}{2}}}{x^{1+\gamma_2}}\ \|\rho_{\gamma_2} f\|_1\,.\end{aligned}$$

Proof. Using twice the Fourier Transform, we get (with $i = 1$ or $i = 3$) e.g.

$$\begin{aligned}\|\big(\mathcal{R}_i f\big)(x)\|_\infty &\le \sum_{n\in\mathbf{Z}} \int_{-\infty}^{\infty}\mathrm{d}k\ \big||k|^{\gamma_2} e^{\Lambda_- x}\big| \int_{-\infty}^{\infty}\mathrm{d}y\ \left|\frac{e^{iky}-1-iky}{|ky|^{\gamma_2}}\right|\ |y|^{\gamma_2}|f_n(y)|,\\ \|\partial_y^m\big(\mathcal{R}_2 f\big)(x)\|_\infty &\le \sum_{n\in\mathbf{Z}} \int_{-\infty}^{\infty}\mathrm{d}k\ \big||k|^{m+\gamma} e^{\Lambda_- x}\big| \int_{-\infty}^{\infty}\mathrm{d}y\ \left|\frac{e^{iky}-1}{|ky|^{\gamma}}\right|\ |y|^{\gamma}|f_n(y)|,\end{aligned}$$

and similar estimates for $\|\partial_y^m\big(\mathcal{R}_2 f\big)(x)\|_2$ and $\|\rho_1\partial_y^m\big(\mathcal{R}_2 f\big)(x)\|_2$. The proof is completed using Lemma A.2 and $\|\partial_y^m \mathcal{R}_2 f\|_1 \le \big(\|\partial_y^m \mathcal{R}_2 f\|_2 \|\rho_1 \partial_y^m \mathcal{R}_2 f\|_2\big)^{\frac{1}{2}}$. □

4.1. The proof of Corollary 1.5

Let $a_{1,1} = -\mathcal{M}_0(\mathcal{I}\mathcal{P}_0 w)$, $a_{1,2} = \int_{\Omega_+} \mathcal{P}_0 Q(x,y)\,\mathrm{d}x\mathrm{d}y$, $\mathbf{a}_{1,1} = (a_{1,1},0,0,0,0,0)$, $\mathbf{a}_{1,2} = (-a_{1,2},0,0,0,0,0)$ and $\mathbf{a}_1 = \mathbf{a}_{1,1}+\mathbf{a}_{1,2}$. Define $u_{L,4}(x) = -K_1(x-x_0)\mathcal{I}\mathcal{P}_0 w$ and $\omega_{L,1}(x) = \partial_y K_1(x-x_0)\mathcal{I}\mathcal{P}_0 w$. We have

$$u(x) - u_{\mathbf{a}_1}(x) = \sum_{i=1}^{6} U_i(x), \qquad \omega(x) - \omega_{\mathbf{a}_1}(x) = \sum_{i=1}^{6} W_i(x)$$

where

$$\begin{aligned}U_1(x) &= u_L(x) - u_{L,4}(x), & U_2(x) &= u_{L,4}(x) - u_{\mathbf{a}_{1,1}}(x-x_0)\\ U_3(x) &= u_{\mathbf{a}_{1,1}}(x-x_0) - u_{\mathbf{a}_{1,1}}(x), & U_4(x) &= u_N(x) - \mathcal{P}_0 u_{N,1}(x),\\ U_5(x) &= \mathcal{P}_0 u_{N,1}(x) - u_{\mathbf{a}_{1,2}}(x-x_0), & U_6(x) &= u_{\mathbf{a}_{1,2}}(x-x_0) - u_{\mathbf{a}_{1,2}}(x)\end{aligned}$$

and a mirror definition of W_i, $i = 1, \ldots, 6$. We complete the proof of Corollary 1.5 by proving that for all $\varepsilon > 0$, there exists a constant C such that

$$\begin{aligned} \|U_i(x)\|_{\infty,1-(1+\varepsilon)\varphi} + \|W_i(x)\|_{\infty,\frac{3}{2}-(1+\varepsilon)\varphi} &\leq C, \\ \|W_i(x)\|_{1,1-(1+\varepsilon)\varphi} + \|\rho_{\beta_0} W_i(x)\|_{2,\frac{5}{4}-\frac{\beta_0}{2}-(1+\varepsilon)\varphi} &\leq C \end{aligned} \tag{4.1}$$

for all $i = 1, \ldots, 6$, $x \geq 2x_0$ and $\frac{1}{2} \leq \beta_0 \leq 1 - 2(1+\varepsilon)\varphi$. The proof of (4.1) with $i = 1$ follows from $W_1 = \mathcal{P}\omega_L$ and Lemma 3.5, 3.6 and A.5. For $i = 2$, it follows from $\|\rho_1 \mathcal{I}\mathcal{P}_0 w\|_1 \leq C\|(0,0,w)\|$ (see Lemma 3.2) and Lemma 4.1 and A.10. For $i = 3$ and $i = 6$, it follows from Lemma A.11. For $i = 4$, it follows from $U_4(x) = \mathcal{P}u_{N,1}(x) + \sum_{i=2}^{6} u_{N,i}(x)$ and Propositions 3.8 to 3.14. Finally, for $i = 5$, it follows from the

Proposition 4.2. *If Q satisfies* (3.4)*, then the estimates* (4.1) *with $i = 5$ are true.*

Remark 4.3. This result follows from similar results of the now classical theory on the nonlinear heat equation (see e.g. [3]). In our case, $\mathbf{a}_{1,2}$ *does not* depend on u, v and ω on the whole domain Ω_+, but only on u and v on the boundary $x = x_0$. Namely, since $Q = -\partial_y R + \partial_x S$, we have

$$\mathcal{P}_0 \int_{\Omega_+} Q(x,y)\,\mathrm{d}x\mathrm{d}y = \mathcal{P}_0 \int_{\Omega_+} \big(\partial_x S(x,y) - \partial_y R(x,y)\big)\,\mathrm{d}x\mathrm{d}y = -\mathcal{M}_0(\mathcal{P}_0 S(x_0)).$$

Proof. Using the relations $\mathcal{P}_0 K_7 \equiv 0$, $\partial_x \mathcal{P}_0 K_8 = -\partial_y K_2$, $\partial_x \mathcal{P}_0 K_2 = \partial_y K_6$, $\mathcal{P}_0 K_2 = 2\partial_y \mathcal{P}_0 K_6 + \partial_y \mathcal{P}_0 K_1$ and $\mathcal{P}_0 K_8 = -\mathcal{P}_0 K_1 - \mathcal{P}_0 K_6$, integrating by parts in $\tilde{x}$, we find the following decompositions

$$U_5(x) = \sum_{i=1}^{5} U_{5,i}(x), \qquad W_5(x) = \sum_{i=1}^{5} W_{5,i}(x), \tag{4.2}$$

with $W_{5,1} = -\partial_y U_{5,1}$, $W_{5,2} = -2\partial_y U_{5,2}$, $W_{5,4}(x) = \mathcal{P}W_5(x) = \mathcal{P}\omega_{N,1}(x)$ and $W_{5,5} = -\partial_y U_{5,4}$, where

$$U_{5,1}(x) = -\mathcal{P}_0 K_1(x-x_0) \int_x^\infty \mathrm{d}z\, Q(z), \qquad U_{5,2}(x) = \mathcal{P}_0 K_6(x-x_0) \int_{x_0}^x \mathrm{d}z\, Q(z),$$

$$U_{5,3}(x) = \mathcal{P}_0 \int_{x_0}^x \mathrm{d}\tilde{x}\, \partial_y K_2(x-\tilde{x}) \int_{\tilde{x}}^x \mathrm{d}z\, Q(z),$$

$$U_{5,4}(x) = -u_{\mathbf{a}_{1,2}}(x-x_0) + \mathcal{P}_0 K_1(x-x_0) \int_{x_0}^\infty \mathrm{d}z\, Q(z),$$

$$W_{5,3}(x) = -\mathcal{P}_0 \int_{x_0}^x \mathrm{d}\tilde{x}\, \partial_y K_6(x-\tilde{x}) \int_{\tilde{x}}^x \mathrm{d}z\, Q(z).$$

Then, for all $x \geq 2x_0$, we have

$$\|U_{5,1}(x)\|_{\infty,1-\varphi} \leq C \sup_{x \geq 2x_0} \langle x \rangle^{1-\varphi} \int_x^\infty \mathrm{d}z\, \langle z \rangle^{-2+\varphi} \leq C,$$

$$\|U_{5,2}(x)\|_{\infty,1} \leq C \sup_{x\geq 2x_0} \frac{\langle x\rangle}{(x-x_0)} \int_{x_0}^{x} \mathrm{d}z\, \langle z\rangle^{-\frac{3}{2}+\varphi} \leq C.$$

With similar arguments, we conclude that

$$\|W_{5,1}(x)\|_{\infty,\frac{3}{2}-\varphi} + \|W_{5,1}(x)\|_{1,1-\varphi} + \|\rho_{\beta_0} W_{5,1}(x)\|_{2,\frac{5}{4}-\frac{\beta_0}{2}-\varphi} \leq C,$$
$$\|W_{5,2}(x)\|_{\infty,\frac{3}{2}} + \|W_{5,2}(x)\|_{1,\frac{3}{2}} + \|\rho_{\beta} W_{5,2}(x)\|_{2,\frac{5}{4}-\frac{\beta}{2}-\varphi} \leq C.$$

Note that the inequality on $\|\rho_{\beta_0} W_{5,1}(x)\|_{2,\frac{5}{4}-\frac{\beta_0}{2}-\varphi}$ is only valid if $\beta_0 < \frac{3}{2} - 2\varphi$. We then have (see Lemma 4.4 below for the definition of $D[\,\cdot\,](x,x_0)$ and related estimates)

$$\|U_{5,3}(x)\|_{\infty} \leq C\langle x\rangle^{\frac{1}{2}} D\begin{bmatrix} 2,\frac{3}{2}-\varphi \\ \frac{3}{2},2-\varphi \end{bmatrix}(x,x_0) \leq C\langle x\rangle^{-1+\varphi},$$

$$\|W_{5,3}(x)\|_{\infty} \leq C\langle x\rangle^{\frac{1}{4}} D\begin{bmatrix} \frac{7}{4},2-\varphi \\ \frac{7}{4},2-\varphi \end{bmatrix}(x,x_0) \leq C\langle x\rangle^{-\frac{3}{2}+\varphi},$$

$$\|W_{5,3}(x)\|_{1} \leq C\langle x\rangle^{\frac{1}{4}} D\begin{bmatrix} \frac{7}{4},\frac{3}{2}-\varphi \\ \frac{7}{4},\frac{3}{2}-\varphi \end{bmatrix}(x,x_0) \leq C\langle x\rangle^{-1+\varphi}.$$

Similarly, we find $\|\rho_\beta W_{5,3}(x)\|_2 \leq C\langle x\rangle^{-\frac{5}{4}+\frac{\beta}{2}+\varphi}$. By Lemma 3.7, we then have

$$\|W_{5,4}(x)\|_{\infty} \leq CB\begin{bmatrix} 1,\frac{3}{2}-\varphi,0 \\ \frac{1}{2},2-\varphi,0 \end{bmatrix}(x,x_0) \leq C\langle x\rangle^{-\frac{3}{2}+\varphi},$$

$$\|W_{5,4}(x)\|_{1} \leq CB\begin{bmatrix} \frac{1}{2},1,0 \\ \frac{1}{2},1,0 \end{bmatrix}(x,x_0) \leq C\langle x\rangle^{-1+\varphi},$$

$$\begin{aligned}\|\rho_\beta W_{5,4}(x)\|_{2} &\leq C\Big(B\begin{bmatrix} \frac{3}{4}-\frac{\beta}{2},\frac{3}{2}-\varphi,0 \\ \frac{3}{4}-\frac{\beta}{2},\frac{3}{2}-\varphi,0 \end{bmatrix}(x,x_0) + B\begin{bmatrix} \frac{1}{2},\frac{7}{4}-\frac{\beta}{2}-\varphi,0 \\ \frac{1}{2},\frac{7}{4}-\frac{\beta}{2}-\varphi,0 \end{bmatrix}(x,x_0)\Big) \\ &\leq C\langle x\rangle^{-\frac{5}{4}+\frac{\beta}{2}+\varphi}.\end{aligned}$$

Finally, using Lemma 4.1 and A.10, and $x \geq 2x_0$, we get that

$$\|U_{5,4}(x)\|_{\infty} \leq C\langle x\rangle^{-\frac{1+\gamma}{2}} \int_{x_0}^{\infty} \mathrm{d}z\, \|\rho_\gamma Q(z)\|_1,$$

$$\|W_{5,5}(x)\|_{\infty} \leq C\langle x\rangle^{-1-\frac{\gamma}{2}} \int_{x_0}^{\infty} \mathrm{d}z\, \|\rho_\gamma Q(z)\|_1,$$

$$\|W_{5,5}(x)\|_{1} \leq C\langle x\rangle^{-\frac{3+\gamma}{4}} \int_{x_0}^{\infty} \mathrm{d}z\, (\|\rho_1 Q(z)\|_1 \|\rho_\gamma Q(z)\|_1)^{\frac{1}{2}},$$

$$\|\rho_{\beta_0} W_{5,5}(x)\|_{2} \leq C\langle x\rangle^{-\frac{3}{4}-\frac{\gamma}{2}(1-\beta_0)} \int_{x_0}^{\infty} \mathrm{d}z\, \|\rho_\gamma Q\|_1^{1-\beta_0} \|\rho_1 Q\|_1^{\beta_0}, \tag{4.3}$$

for any $0 \leq \gamma \leq 1$ (we used $\|\rho_{\beta_0} f\|_p \leq \|f\|_p^{1-\beta_0} \|\rho_1 f\|_p^{\beta_0}$ in (4.3)). Then, for any $\gamma_1 \leq 1$, $\gamma_2 \leq 1$, $\gamma_3 \leq 1$ and $\sigma > \frac{1}{2}$, we have (using $\|f\|_1 \leq C_\sigma \|\rho_\sigma f\|_2$ for $\sigma > \frac{1}{2}$)

$$\int_{x_0}^{\infty} \mathrm{d}z\, \|\rho_{\gamma_1} Q(z)\|_1 \leq C \int_{x_0}^{\infty} \mathrm{d}z\, \langle z\rangle^{\varphi+\frac{\gamma_1+\sigma}{2}-\frac{7}{4}},$$

$$\int_{x_0}^{\infty} \mathrm{d}z \, (\|\rho_1 Q(z)\|_1 \|\rho_{\gamma_2} Q(z)\|_1)^{\frac{1}{2}} \le C \int_{x_0}^{\infty} \mathrm{d}z \, \langle z \rangle^{\varphi + \frac{\gamma_2}{4} + \frac{\sigma}{2} - \frac{3}{2}},$$

$$\int_{x_0}^{\infty} \mathrm{d}z \, \|\rho_{\gamma_3} Q(z)\|_1^{1-\beta_0} \|\rho_1 Q(z)\|_1^{\beta_0} \le C \int_{x_0}^{\infty} \mathrm{d}z \, \langle z \rangle^{\varphi + \frac{\beta_0}{2} + \frac{\gamma_3(1-\beta_0)}{2} + \frac{\sigma}{2} - \frac{7}{4}} .$$

Choosing $\gamma_1 = 1-2(1+\varepsilon)\varphi$, $\gamma_2 = 1-4(1+\varepsilon)\varphi$, $\gamma_3 = 1-2\big(\frac{1+\varepsilon}{1-\beta_0}\big)\varphi$ and $\sigma = \frac{1}{2}+\varepsilon\varphi$ with $\varepsilon > 0$ completes the proof. ☐

Lemma 4.4. *Let $0 \le p_1, q_2 < 2$, and $p_2, q_1 \ge 0$, then there exists a constant C such that*

$$D\begin{bmatrix} p_2, q_2 \\ p_1, q_1 \end{bmatrix}(x, x_0) \equiv \int_{x_0}^{x} \mathrm{d}\tilde{x} \int_{\tilde{x}}^{x} \mathrm{d}z \ \min\Big(\frac{\langle z \rangle^{-q_1}}{(x-\tilde{x})^{p_1}}, \ \frac{\langle z \rangle^{-q_2}}{(x-\tilde{x})^{p_2}}\Big)$$
$$\le C\big(\langle x \rangle^{2-p_1-q_1} + \langle x \rangle^{2-p_2-q_2}\big).$$

for all $x \ge 2x_0 \ge 2$.

Proof. The proof follows at once (see also the proof of Lemma 3.7) from

$$D\begin{bmatrix} p_2, q_2 \\ p_1, q_1 \end{bmatrix}(x, x_0) \le \frac{C}{(x-x_0)^{p_2}} \int_{x_0}^{\frac{x+x_0}{2}} \mathrm{d}\tilde{x} \int_{\tilde{x}}^{x} \frac{\mathrm{d}z}{\langle z \rangle^{q_2}} + C\langle x \rangle^{-q_1} \int_{\frac{x+x_0}{2}}^{x} \mathrm{d}\tilde{x} \ (x-\tilde{x})^{1-p_1}. \quad \square$$

5. Refined asymptotics

To complete the asymptotic description of solution of (1.1), we now prove Corollary 1.6. Since the asymptotic description of ω is already proved in Corollary 1.5, it only remains to prove (1.18). As in Section 4, to avoid the proliferation of symbols, in this section, the letter C stands for a constant which may change its value from instance to instance and depends on x_0, $\|(\nu, \mathcal{H}\nu, w)\|$, $|||(u, v, \omega)|||$, $\|\rho_1(u(x_0)^2 + v(x_0)^2)\|_2$, $\|\rho_{\frac{1}{2}-(1+\varepsilon)\varphi} \mathcal{S}\nu\|_1$ and $\|\rho_{\frac{1}{2}-(1+\varepsilon)\varphi} \mathcal{S}\mathcal{H}\nu\|_1$. We again note that we need only prove the estimates for $x \ge 2x_0$. The proof of Corollary 1.6 stands on three pillars, the partial description of Corollary 1.5, Lemma 4.1 and its equivalent on K_{12}, K_{13} and K_0:

Lemma 5.1. *Let $0 \le \gamma \le 1$ and f satisfying $\|\langle y \rangle^{\gamma} f\|_1 < \infty$. Then for all $m \ge 0$, there exist constants C_γ such that*

$$\|\partial_y^m K_{12}(x)(f - \mathcal{M}_0(f))\|_\infty + \|\partial_y^m K_{13}(x)(f - \mathcal{M}_0(f))\|_\infty \le C_\gamma x^{-1-m-\gamma} \ \|\rho_\gamma f\|_1,$$
$$\|\partial_y^m K_0(x)(f - \mathcal{M}_0(f))\|_\infty + \|\partial_y^m \mathcal{H} K_0(x)(f - \mathcal{M}_0(f))\|_\infty \le C_\gamma x^{-1-m-\gamma} \ \|\rho_\gamma f\|_1 .$$

Proof. The proof follows along the same lines as that of Lemma 4.1, e.g.

$$\|\partial_y^m K_{12}(x)(f - \mathcal{M}_0(f))\|_\infty \le C_\gamma \sum_{n \in \mathbf{Z}} \int_{-\infty}^{\infty} \mathrm{d}k \ \Big| |k|^{m+\gamma} e^{-|k|x} \Big| \ \int_{-\infty}^{\infty} \mathrm{d}y \ |y|^\gamma |f_n(y)|.$$

The other estimates are similar. □

The proof of Corollary 1.6 will now be split in the two following subsections. Using the first order results on ω and u, we will prove the v estimate in (1.18) in a first round of estimates in Subsection 5.1. We will then use the v estimate to prove the u estimate in a second round of estimates in Subsection 5.2. In principle, this 'ping-pong' strategy could be systematically used to get higher order asymptotic developments.

5.1. The 'v' component

We now prove the asymptotic description of the 'v' component:

Proposition 5.2. *Let $a_{1,1} = -\mathcal{M}_0(\mathcal{I}\mathcal{P}_0 w)$, $a_{1,2} = \int_{\Omega_+}\mathcal{P}_0 Q(x,y)\mathrm{d}x\mathrm{d}y$, $a_{2,1} = \mathcal{M}_0(\mathcal{S}\nu)$, $a_3 = \mathcal{M}_0(\mathcal{S}\mathcal{H}\nu)$ and $\mathbf{a}_2 = (a_{1,1} - a_{1,2}, a_{2,1} + a_{1,2}, a_3, 0, 0, 0)$. Then for all $\varepsilon > 0$ and $\frac{1}{2} \leq \beta_0 \leq 1 - 2(1+\varepsilon)\varphi$, there exists a constant C such that*

$$\|u - u_{\mathbf{a}_2}\|_{\infty,1-(1+\varepsilon)\varphi} + \|v - v_{\mathbf{a}_2}\|_{\infty,\frac{3}{2}-(1+\varepsilon)\varphi} + +\|\omega - \omega_{\mathbf{a}_2}\|_{\infty,\frac{3}{2}-(1+\varepsilon)\varphi} \leq C,$$
$$\|\omega - \omega_{\mathbf{a}_2}\|_{1,1-(1+\varepsilon)\varphi} + \|\rho_{\beta_0}(\omega - \omega_{\mathbf{a}_2})\|_{2,\frac{5}{4}-\frac{\beta_0}{2}-(1+\varepsilon)\varphi} \leq C.$$

Proof. We first note that we only need to prove $\|v - v_{\mathbf{a}_2}\|_{\infty,\frac{3}{2}-(1+\varepsilon)\varphi} \leq C$, as the other estimates follow immediately from easy algebra and Corollary 1.5. Let $\mathbf{a}_{1,2} = (-a_{1,2}, 0, 0, 0, 0, 0)$, $\mathbf{a}_{1,3} = (0, a_{1,2}, 0, 0, 0, 0)$ and $\mathbf{a}_L = (a_{1,1}, a_{2,1}, a_3, 0, 0, 0)$. Define $V_5(x) = v_{N,1}(x) + v_{N,3}(x) + v_{N,4}(x) + v_{N,5}(x) + \omega_{N,2}(x) + \omega_{N,3}(x) + \omega_{N,4}(x)$, $V_6(x) = v_{N,6}(x)$, $V_7(x) = \omega_{N,1}(x) - v_{\mathbf{a}_{1,2}}(x)$, $V_8(x) = v_{N,2}(x) - v_{\mathbf{a}_{1,3}}(x - x_0)$, $V_9(x) = v_L(x) - v_{\mathbf{a}_L}(x)$ and $V_{10}(x) = v_{\mathbf{a}_2}(x - x_0) - v_{\mathbf{a}_2}(x)$. We have

$$v_N(x) - v_{\mathbf{a}_2}(x) = \sum_{i=5}^{10} V_i(x).$$

The proof is completed once we show that for $i = 5, \ldots, 10$ and all $x \geq 2x_0$, it holds

$$\|V_i(x)\|_{\infty,\frac{3}{2}-(1+\varepsilon)\phi} \leq C. \tag{5.1}$$

The $i = 5$ estimate follows from Propositions 3.9 to 3.14, that with $i = 10$ follows from Lemma A.11, that with $i = 7$ follows from Proposition 4.2 above, while those with $i = 6$, $i = 8$ and $i = 9$ follow from the Lemmas 5.4, 5.5 and 5.6 below. □

Before turning to the technical results needed to prove Proposition 5.2, we note that it follows from Proposition 5.2 that $\mathcal{P}_0 Q$ also has nice asymptotic properties as shows the

Corollary 5.3. *Let* $\mathbf{a}_2 = (a_1, a_2, a_3, 0, 0, 0)$ *be given by Proposition 5.2, and define* $\Delta Q = \mathcal{P}_0(Q - Q_{\mathbf{a}_2})$ *where*

$$Q_{\mathbf{a}_2}(x,y) = \tfrac{a_1^2}{x^2} f_1\big(\tfrac{y}{\sqrt{x}}\big)^2 + \tfrac{a_1 \mathcal{P}_0 a_3}{x^2} f_1\big(\tfrac{y}{\sqrt{x}}\big), \tag{5.2}$$

then for all $\varepsilon > 0$*, we have*

$$\|\!|\Delta Q|\!\|_{\infty, \frac{5}{2}-(1+\varepsilon)\varphi} + \|\!|\Delta Q|\!\|_{1, 2-(1+\varepsilon)\varphi} + \|\!|\rho_\gamma \Delta Q|\!\|_{1, \frac{9}{4}-\gamma-2\varphi(1+\frac{3\varepsilon}{4})} \le C \tag{5.3}$$

for all $\frac{1}{2} \le \gamma \le \frac{5}{4} - 2\varphi(1+\varepsilon)$.

Proof. Let $\tilde{Q}_{\mathbf{a}_2} = v_{\mathbf{a}_2}\omega_{\mathbf{a}_2}$. We have $\Delta Q = \mathcal{P}_0 \Delta Q_1 + \Delta Q_2$, where $\Delta Q_1 = Q - \tilde{Q}_{\mathbf{a}_2}$ and $\Delta Q_2(x,y) = \mathcal{P}_0 \tilde{Q}_{\mathbf{a}_2}(x,y) - Q_{\mathbf{a}_2}(x,y)$ is given by

$$\Delta Q_2(x,y) = \tfrac{a_1 \mathcal{P}_0 a_2}{x^{\frac{5}{2}}} f_2\big(\tfrac{y}{\sqrt{x}}\big) g_0\big(\tfrac{y}{x}\big) - \tfrac{a_1 \mathcal{P}_0 a_3}{x^3} f_3\big(\tfrac{y}{\sqrt{x}}\big) g_0\big(\tfrac{y}{x}\big).$$

We first note that ΔQ_2 satisfies (5.3), then by Proposition 5.2, $\|\!|\Delta Q_1|\!\|_{\infty, \frac{5}{2}-(1+\varepsilon)\varphi} \le C$ and $\|\!|\Delta Q_1|\!\|_{1, 2-(1+\varepsilon)\varphi} \le C$. Now let $\frac{1}{2} \le \gamma \le \frac{5}{4} - 2\varphi(1+\varepsilon)$ and define $\varepsilon_1 = 1 - \frac{1}{2\gamma}(1-(4+\varepsilon)\varphi)$, $\beta_0 = (1-\varepsilon_1)\gamma + \frac{1+\varepsilon}{2}\varphi$, $\Delta v(x) = v(x) - v_{\mathbf{a}_1}(x)$ and $\Delta\omega(x) = \omega(x) - \omega_{\mathbf{a}_1}(x)$. By hypothesis, we have $\gamma\varepsilon_1 \le 1$ and $0 \le \beta_0 \le 1 - 2(1+\varepsilon)\varphi$, so that

$$\begin{aligned} \|\rho_\gamma \Delta Q_1(x)\|_1 &\le \|\Delta v(x)\|_\infty \|\rho_\gamma \omega(x)\|_1 + \|\rho_{\varepsilon_1\gamma} v_{\mathbf{a}_1}(x)\|_\infty \|\rho_{\beta_0} \Delta\omega(x)\|_2 \\ &\le C\langle x\rangle^{-2+\frac{\gamma}{2}+2\varphi(1+\frac{3\varepsilon}{4})} + C\langle x\rangle^{-\frac{9}{4}+\gamma+2\varphi(1+\frac{3\varepsilon}{4})}. \end{aligned}$$

This completes the proof since $\gamma \ge \frac{1}{2}$. $\square$

We now prove the technical results needed to finish the proof of Proposition 5.2. For further reference, we also include the 'u' component counterpart of each result, as they are proved with very similar arguments.

Lemma 5.4. *Let* $\mathbf{a}_1 = (a_1, 0, 0, 0, 0, 0)$ *be given by Corollary 1.5, and define* $P_{\mathbf{a}_1}(x) = u_{\mathbf{a}_1}(x)\omega_{\mathbf{a}_1}(x)$*, then there exists a constant* C *such that* $\|\!|\mathcal{L}_1 R|\!\|_{\infty, \frac{3}{2}-\varphi} + \|\!|\mathcal{L}_2 R|\!\|_{\infty, \frac{3}{2}-\varphi} + \|\!|\mathcal{L}_1 S - \mathcal{I}P_{\mathbf{a}_1}|\!\|_{\infty, \frac{3}{2}-(1+\varepsilon)\varphi} + \|\!|\mathcal{L}_2 S|\!\|_{\infty, \frac{3}{2}-(1+\varepsilon)\varphi} \le C.$

Proof. By Corollary 1.5, we have

$$\|S(x) - \mathcal{I}P_{\mathbf{a}_1}(x)\|_\infty \le \|v(x)\|_\infty^2 + \|u(x) - u_{\mathbf{a}_1}(x)\|_\infty \|u(x) + u_{\mathbf{a}_1}(x)\|_\infty.$$

This proves $\|\!|\mathcal{L}_1 S - \mathcal{I}P_{\mathbf{a}_1}|\!\|_{\infty, \frac{3}{2}-(1+\varepsilon)\varphi} \le C$. The other estimates follow easily from this last one, $\mathcal{P}P_{\mathbf{a}_1}(x) = 0$, $\mathcal{P}_0\mathcal{L}_1 = \mathbb{1}$, $\mathcal{P}_0\mathcal{L}_2 = 0$ and Lemma A.3. $\square$

Lemma 5.5. *There exists a constant* C *such that for all* $x \ge 2x_0$*, it holds*

$$\|u_{N,2}(x) - u_{\mathbf{a}_{1,3}}(x - x_0)\|_{\infty, \frac{3}{2}-(1+\varepsilon)\varphi} + \|v_{N,2}(x) - v_{\mathbf{a}_{1,3}}(x - x_0)\|_{\infty, \frac{3}{2}-(1+\varepsilon)\varphi} \le C.$$

Proof. The proof is very similar to the one of Proposition 4.2. We define $T_1(x) = K_{12}(x - x_0)\int_x^\infty \mathrm{d}z\ Q(z)$, $T_2(x) = \int_{x_0}^x \mathrm{d}\tilde{x}\ \partial_y K_{13}(x - \tilde{x})\int_{\tilde{x}}^x \mathrm{d}z\ Q(z)$, $T_3(x) =$

$-\mathcal{P}K_{12}(x-x_0)\int_{x_0}^{\infty}\mathrm{d}z\; Q(z)$ and $T_4(x) = -\mathcal{P}_0 K_{12}(x-x_0)\int_{x_0}^{\infty}\mathrm{d}z\; Q(z) - u_{\mathbf{a}_{1,3}}(x-x_0)$. Since $\partial_x K_{12}(x) = \partial_y K_{13}(x)$, after integration by parts, we have

$$u_{N,2}(x) - u_{\mathbf{a}_{1,3}}(x-x_0) = T_1(x) + T_2(x) + T_3(x) + T_4(x).$$

Since $\|K_{12}\|_{\frac{1}{2\varepsilon\varphi},\{0,1-2\varepsilon\varphi\}} + \|K_{13}\|_{\frac{1}{2\varepsilon\varphi},\{0,1-2\varepsilon\varphi\}} \le C$ and $|\!|\!|Q|\!|\!|_{\frac{1}{1-2\varepsilon\varphi},\frac{3}{2}-(1-\varepsilon)\varphi} \le C$, we have for all $x \ge 2x_0$ (using Lemma 4.4 in the second inequality) that

$$\begin{aligned}
\|T_1(x)\|_\infty &\le C(x-x_0)^{-1}\int_x^\infty \mathrm{d}z\; z^{-\frac{3}{2}+\varphi} \le C\langle x\rangle^{-\frac{3}{2}+\varphi} \\
\|T_2(x)\|_\infty &\le CD\begin{bmatrix}2-2\varepsilon\varphi,\frac{3}{2}-\varphi(1-\varepsilon)\\ 2-2\varepsilon\varphi,\frac{3}{2}-\varphi(1-\varepsilon)\end{bmatrix}(x,x_0) \le C\langle x\rangle^{-\frac{3}{2}+(1+\varepsilon)\varphi}, \\
\|T_3(x)\|_\infty &\le C\langle x\rangle^{-2+\varphi}\int_{x_0}^\infty \mathrm{d}z\; \langle z\rangle^{-\frac{3}{2}+\varphi} \le C\langle x\rangle^{-2+\varphi}, \\
\|T_4(x)\|_\infty &\le C\langle x\rangle^{-\frac{3}{2}+(1+\varepsilon)\varphi}\int_{x_0}^\infty \mathrm{d}z\; \langle z\rangle^{-1-\frac{1}{4}(1-2\varphi)},
\end{aligned}$$

since $\|\rho_{\frac{1}{2}-(1+\varepsilon)\varphi}Q(x)\|_1 \le C\|\rho_{1-\varphi}Q(x)\|_2 \le C\langle x\rangle^{-1-\frac{1}{4}(1-2\varphi)}$ and $\mathcal{P}_0K_{12} = \mathcal{P}_0K_0$. The proof of $\|u_{N,2}(x) - u_{\mathbf{a}_{1,3}}(x-x_0)\|_{\infty,\frac{3}{2}-(1+\varepsilon)\varphi} \le C$ is completed using $\varphi < \frac{1}{2}$. That of $\|v_{N,2}(x) - u_{\mathbf{a}_{1,3}}(x-x_0)\|_{\infty,\frac{3}{2}-(1+\varepsilon)\varphi} \le C$ being very similar, we omit the details. □

Lemma 5.6. *Let* $\mathbf{a}_{2,1} = (a_{1,1}, a_{2,1}, a_3, a_{2,4}, 0, 0)$, *where* $a_{1,1} = -\mathcal{M}_0(\mathcal{I}\mathcal{P}_0 w)$, $a_{2,1} = \mathcal{M}_0(\mathcal{S}\nu)$, $a_3 = \mathcal{M}_0(\mathcal{S}\mathcal{H}\nu)$, $a_{2,4} = -\mathcal{M}_1(\mathcal{I}\mathcal{P}_0 w)$. *There exists a constant* C *such that for all* $x \ge 2x_0$, *it holds*

$$\|u_L(x) - u_{\mathbf{a}_{2,1}}(x)\|_{\infty,\frac{5}{4}-(1+\varepsilon)\varphi} + \|v_L(x) - v_{\mathbf{a}_{2,1}}(x)\|_{\infty,\frac{3}{2}-(1+\varepsilon)\varphi} \le C. \tag{5.4}$$

Proof. Let $u_{L,4}(x) = -K_1(x-x_0)\mathcal{I}\mathcal{P}_0 w$, $v_{L,4}(x) = \partial_y K_1(x-x_0)\mathcal{I}\mathcal{P}_0 w$, $u_{L,5}(x) = K_0(x-x_0)\mathcal{S}\nu$, $u_{L,6}(x) = -\mathcal{H}K_0(x-x_0)\mathcal{S}\mathcal{H}\nu$, $v_{L,5}(x) = K_0(x-x_0)\mathcal{S}\mathcal{H}\nu$, $v_{L,6}(x) = \mathcal{H}K_0(x-x_0)\mathcal{S}\nu$, $U_4(x) = \mathcal{P}K_1(x-x_0)\mathcal{L}_u w$, $V_4(x) = \mathcal{P}K_1(x-x_0)\mathcal{L}_v w$ and $U_5(x) = \mathcal{P}_0 K_1(x-x_0)(\mathcal{L}_u + \mathcal{I})w$. Since $\nu = \mathcal{S}\nu - \mathcal{H}\mathcal{S}\mathcal{H}\nu$ and $\mathcal{H}\nu = \mathcal{S}\mathcal{H}\nu + \mathcal{H}\mathcal{S}\nu$, we have

$$u_L(x) - u_{\mathbf{a}_{2,1}}(x) = \sum_{i=1}^{5} U_i(x), \qquad v_L(x) - v_{\mathbf{a}_{2,1}}(x) = \sum_{i=1}^{4} V_i(x),$$

where $U_1(x) = u_{L,5}(x) + u_{L,6}(x) - u_{\mathbf{a}_{2,2}}(x-x_0)$ for $\mathbf{a}_{2,2} = (0, a_{2,1}, a_{3,1}, 0, 0, 0)$, $U_2(x) = u_{L,4}(x) - u_{\mathbf{a}_{2,3}}(x-x_0)$ for $\mathbf{a}_{2,3} = (a_{1,1}, 0, 0, a_{4,1}, 0, 0)$, $U_3(x) = u_{\mathbf{a}_{2,1}}(x-x_0) - u_{\mathbf{a}_{2,1}}(x)$ and the same definitions for V_i for $i = 1, 2, 3$, mutatis mutandis. Using Lemmas A.11, 4.1, 5.1 and A.10, and that by Lemma 3.2 we have $\|\rho_\gamma \mathcal{I}\mathcal{P}_0 w\|_1 \le C$ for all $\gamma \le \frac{3}{2} - 2(1+\varepsilon)\varphi$, we conclude that for $x \ge 2x_0$ and $i = 1, 2, 3$, it holds

$$\|U_i(x)\|_{\infty,\frac{3}{2}-(1+\varepsilon)\varphi} + \|V_i(x)\|_{\infty,\frac{3}{2}-(1+\varepsilon)\varphi} \le C.$$

We then note that by Lemma A.5 and 3.5, for all $m \geq 0$ and $x \geq 2x_0$, we have

$$\|U_4(x)\|_{\infty,m} + \|V_4(x)\|_{\infty,m} \leq C_m \tag{5.5}$$

since $\|\mathcal{P}K_1(x)\|_1$ decays exponentially as $x \to \infty$. Easy algebra also shows that

$$\sup_{x \geq 2x_0} \|U_5(x)\|_{\infty,\frac{3}{2}} \leq \sup_{x \geq 2x_0} C\langle x\rangle^{\frac{3}{2}} \|\partial_y^2 K_1(x-x_0)\|_\infty \|\mathcal{I}(\mathcal{I}(\mathcal{L}_u + \mathcal{I})w)\|_1 .$$

The proof is completed using $\|\mathcal{I}(\mathcal{I}(\mathcal{L}_u + \mathcal{I})w)\|_1 \leq C\|\mathcal{I}w\|_1$. □

5.2. Nonlinear terms, the 'u' component

To complete the proof of Corollary 1.6, we first note that since $u = u_L + u_N$ and Lemma 5.6 already gives the detailed asymptotics of u_L, we need only give the asymptotic description of u_N. To do so, let $\mathbf{a}_2 = (a_1, a_2, a_3, 0, 0, 0)$ be given by Proposition 5.2, $a_{1,2} = \int_{\Omega_+} \mathcal{P}_0 Q(x,y)\,\mathrm{d}x\mathrm{d}y$, $a_{4,3} = \int_{\Omega_+} y\mathcal{P}_0(Q(x,y) - Q_{\mathbf{a}_2}(x,y))\mathrm{d}x\mathrm{d}y$ (this quantity is bounded by Corollary 5.3), $\mathbf{a}_{1,3} = (0, a_{1,2}, 0, 0, 0, 0)$,

$$\begin{aligned}
\mathbf{a}_5 &= (0,0,0,-\int_{\mathbf{R}} \mathcal{P}_0 u(x_0,y)v(x_0,y)\,\mathrm{d}y, a_1^2, 0),\\
\mathbf{a}_6 &= (-a_{1,2}, 0, 0, \ln(x_0)a_1\mathcal{P}_0 a_3 + a_{4,3}, 0, a_1\mathcal{P}_0 a_3)
\end{aligned}$$

and $\mathbf{a}_3 = \mathbf{a}_{1,3} + \mathbf{a}_5 + \mathbf{a}_6$. We complete the proof of Corollary 1.6 by proving that

$$\|u_N(x) - u_{\mathbf{a}_3}(x)\|_{\infty,\frac{9}{8}-(1+\varepsilon)\varphi} \leq C.$$

We write $u_N(x) - u_{\mathbf{a}_3}(x) = \sum_{i=6}^{11} U_i(x)$, where $U_6(x) = u_{N,6}(x) - \mathcal{I}P_{\mathbf{a}_1}(x)$ for $P_{\mathbf{a}_1}(x) = u_{\mathbf{a}_1}(x)\omega_{\mathbf{a}_1}(x)$, $\mathbf{a}_1$ is given by Corollary 1.5, $U_7(x) = u_{N,2}(x) - u_{\mathbf{a}_{1,3}}(x-x_0)$,

$$\begin{aligned}
U_8(x) &= \mathcal{P}_0 u_{N,3}(x) + 2\mathcal{I}P_{\mathbf{a}_1}(x) - u_{\mathbf{a}_5}(x),\\
U_9(x) &= \mathcal{P}_0 u_{N,1} - \mathcal{I}P_{\mathbf{a}_1}(x) - u_{\mathbf{a}_6}(x),
\end{aligned} \tag{5.6}$$

$U_{10}(x) = u_{\mathbf{a}_{1,3}}(x-x_0) - u_{\mathbf{a}_{1,3}}(x)$ and $U_{11}(x) = \mathcal{P}(u_{N,1}(x) + u_{N,3}(x)) + u_{N,4}(x) + u_{N,5}(x)$. Then, for $i = 6, \ldots, 11$, these functions satisfy for all $x \geq 2x_0$ the estimate

$$\|U_i(x)\|_{\infty,\sigma_i} \leq C, \tag{5.7}$$

with $\min_{i=6,11} \sigma_i = \frac{9}{8} - (1+\varepsilon)\varphi$. The estimate for $i = 6$, $i = 7$ and $i = 10$ follows with $\sigma_i = \frac{3}{2} - (1+\varepsilon)\varphi$ from Lemmas 5.5, A.11 and 5.4 above. For $i = 11$, it follows from Propositions 3.9 to 3.14 with $\sigma_{11} = \frac{3}{2} - (1+\varepsilon)\varphi$. Finally, we will prove in Propositions 5.7 and 5.8 below that the estimate holds with $\sigma_8 = \frac{5}{4} - \varepsilon\varphi$ for $i = 8$ and with $\sigma_9 = \frac{9}{8} - (1+\varepsilon)\varphi$ for $i = 9$. This completes the proof of Corollary 1.6.

Proposition 5.7. *Let U_8 be defined in (5.6) and assume that $\|\rho_1(u(x_0)^2 + v(x_0)^2)\|_2$ is bounded, then for all $\varepsilon > 0$, there exists a constant C such that for all $x \geq 2x_0$, it holds $\|U_8(x)\|_{\infty,\frac{5}{4}-\varepsilon\varphi} \leq C$.*

Proof. Let $\mathbf{a}_{5,1} = (0,0,0,-\int_{\mathbf{R}} \mathcal{P}_0 u(x_0,y)v(x_0,y)\,\mathrm{d}y,0,0)$, $\mathbf{a}_{5,2} = (0,0,0,0,a_1^2,0)$, $S_1(x) = \frac{1}{2}v(x)^2$ and $S_2(x) = -\frac{1}{2}(u(x)-u_{\mathbf{a}_1}(x))(u(x)+u_{\mathbf{a}_1}(x))$. Using $P(x) - P_{\mathbf{a}_1}(x) = \partial_x R(x) + \partial_y(S_1(x)+S_2(x))$, we get $U_8(x) = \sum_{i=1,8} U_{8,i}(x)$ where

$$
\begin{aligned}
U_{8,1}(x) &= \mathcal{J}_1[\partial_y K_c, P_{\mathbf{a}_1}](x) + 2\mathcal{I}P_{\mathbf{a}_1}(x)\\
U_{8,2}(x) &= \mathcal{P}_0\mathcal{J}_{1,2}[K_2, P-P_{\mathbf{a}_1}](x)\\
U_{8,3}(x) &= \mathcal{P}_0\mathcal{J}_1[K_2-\partial_y K_c, P_{\mathbf{a}_1}](x)\\
U_{8,4}(x) &= \mathcal{P}_0 K_2(x-x_0)R(x_0) + \mathcal{P}_0\mathcal{J}_{1,1}[K_2,\partial_x R](x)\\
U_{8,5}(x) &= \mathcal{P}_0\mathcal{J}_{1,2}[\partial_y K_2, S_1](x), \qquad U_{8,6}(x) = \mathcal{P}_0\mathcal{J}_{1,2}[\partial_y K_2, S_2](x)
\end{aligned}
$$

$U_{8,7}(x) = -\mathcal{P}_0 K_2(x-x_0)R(x_0) - u_{\mathbf{a}_{5,1}}(x-x_0)$ and $U_{8,8}(x) = u_{\mathbf{a}_{5,1}}(x-x_0) - u_{\mathbf{a}_{5,1}}(x)$, where

$$
\begin{aligned}
\mathcal{J}_{1,1}[K,f](x) &= \int_{x_0}^{\frac{x+x_0}{2}} \mathrm{d}\tilde{x}\; K(x-\tilde{x})f(\tilde{x}),\\
\mathcal{J}_{1,2}[K,f](x) &= \int_{\frac{x+x_0}{2}}^{x} \mathrm{d}\tilde{x}\; K(x-\tilde{x})f(\tilde{x}).
\end{aligned}
\tag{5.8}
$$

We first note that the Fourier transform $\hat{T}(x)$ of $U_{8,1}(x)$ reads

$$
\hat{T}(x,k) = \underbrace{\tfrac{ika_1^2}{4}\mathrm{erf}\big(\tfrac{ik\sqrt{x_0}}{\sqrt{2}}\big)e^{-k^2x}}_{\equiv \hat{R}(x,x_0,k)} - \tfrac{1}{\sqrt{x}}\Big(\underbrace{\tfrac{ik\sqrt{x}a_1^2}{4}\mathrm{erf}\big(\tfrac{ik\sqrt{x}}{\sqrt{2}}\big)e^{-k^2x} + \tfrac{a_1^2 e^{-\frac{k^2x}{2}}}{2\sqrt{2\pi}}}_{\equiv \hat{H}(k\sqrt{x})}\Big).
$$

We then note that for $x \geq x_0$, we have $\|R(x)\|_{\infty,\frac{3}{2}} \leq \|\hat{R}(x)\|_{1,\frac{3}{2}} \leq C$, and that the inverse Fourier transform $H(y)$ of $\hat{H}(k)$ satisfies

$$
H''(y) + \tfrac{y}{2}H'(y) + H(y) - \tfrac{a_1^2 e^{-\frac{y^2}{2}}}{8\pi} = 0, \qquad H(0) = \tfrac{a_1^2}{8\pi}, \qquad H'(0) = 0,
$$

whose unique solution is $H(y) = \frac{a_1^2 h(y)}{2}$. This proves that $\|U_{8,1}(x) - u_{\mathbf{a}_{5,2}}(x)\|_{\infty,\frac{3}{2}} \leq C$. Let $p = \frac{1}{1-\varepsilon\varphi}$. Corollary 1.5 implies that $\|P - P_{\mathbf{a}_1}\|_{p,2-(1+\varepsilon)\varphi-\frac{1}{2p}} \leq C$ for all $\varepsilon > 0$. Using Lemma A.10, we get

$$
\begin{aligned}
\|U_{8,2}(x)\|_{\infty,\frac{3}{2}-(1+2\varepsilon)\varphi} &\leq C \sup_{x\geq 2x_0} \langle x\rangle^{\frac{3}{2}-(1+2\varepsilon)\varphi} \int_{\frac{x+x_0}{2}}^{x} \frac{\mathrm{d}\tilde{x}\; \langle\tilde{x}\rangle^{-\frac{3}{2}+(1+\frac{3\varepsilon}{2})\varphi}}{(x-\tilde{x})^{1-\frac{\varepsilon\varphi}{2}}},\\
\|U_{8,3}(x)\|_{\infty} &\leq C \int_{x_0}^{x}\mathrm{d}\tilde{x}\int_{-\infty}^{\infty}\mathrm{d}k\; \frac{\mathcal{P}_0 e^{\Lambda_-(x-\tilde{x})-\frac{k^2}{2}\tilde{x}}(|k|^5(x-\tilde{x})+|k|^3)}{\tilde{x}^{\frac{1}{2}}}\\
&\leq C\int_{x_0}^{x}\mathrm{d}\tilde{x}\,\min\Big(\frac{\langle x-\tilde{x}\rangle^3}{(x-\tilde{x})^5}, \frac{x-\tilde{x}}{\tilde{x}^3} + \frac{1}{\tilde{x}^2}\Big)\tilde{x}^{-\frac{1}{2}},
\end{aligned}
$$

from which we easily get $\|U_{8,2}(x)\|_{\infty,\frac{3}{2}-(1+2\varepsilon)\varphi} + \|U_{8,3}(x)\|_{\infty,\frac{3}{2}} \leq C$. We then note that $\|S_1\|_{1,1-2\varphi} + \|S_2\|_{2,1-\varepsilon\varphi} \leq C$ and $\|\rho_{\frac{1}{2}-\varepsilon\varphi}R(x_0)\|_1 \leq \|\rho_1(u(x_0)^2+v(x_0)^2\|_2 \leq C$. Using these inequalities, Lemma A.11 and 4.1, we get that for all $x \geq 2x_0$, we have $\|U_{8,i}(x)\|_{\infty,\sigma_i} \leq C$ with $\sigma_4 = \frac{3}{2}-\varphi$, $\sigma_5 = \frac{3}{2}-2\varphi$ $\sigma_6 = \frac{5}{4}-\varepsilon\varphi$ $\sigma_7 = \frac{5}{4}-\varepsilon\varphi$

and $\sigma_8 = \frac{3}{2} - \varphi$, and where we used integration by parts, (3.4) and $\|\partial_x K_2\|_\infty \leq \|\partial_y K_6(x-\tilde{x})\|_\infty + \|\partial_y K_7(x-\tilde{x})\|_\infty$ to establish the $U_{8,4}$ estimate. This completes the proof. □

Proposition 5.8. *Let U_9 be defined in* (5.6)*, then for all $\varepsilon > 0$, there exists a constant C such that for all $x \geq 2x_0$, it holds $\|U_9(x)\|_{\infty,\frac{9}{8}-(1+\varepsilon)\varphi} \leq C$.*

Remark 5.9. In view of the corresponding theory on nonlinear heat equations, (see e.g. [3]), using higher moments of Q, i.e. considering $u_{N,1} - u_{\mathbf{a}_3}$ with

$$\mathbf{a}_3 = (\mathcal{P}_0 \int_{\Omega_+} Q(x,y)\, \mathrm{d}x\mathrm{d}y, 0, 0, \mathcal{P}_0 \int_{\Omega_+} yQ(x,y)\, \mathrm{d}x\mathrm{d}y, 0, 0)$$

instead of $u_{N,1} - u_{\mathbf{a}_{1,3}}$ should improve the results of Proposition 4.2. We are forced to work with the first moment of $Q - Q_{\mathbf{a}_2}$ because $\int_{\Omega_+} y\mathcal{P}_0 Q(x,y)\, \mathrm{d}x\mathrm{d}y$ is infinite in general[6].

Proof of Proposition 5.8. We first define $\Delta Q(x) = \mathcal{P}_0(Q(x) - Q_{\mathbf{a}_2}(x))$,

$$a_{1,3} = \int_{\Omega_+} \Delta Q(x,y)\mathrm{d}x\mathrm{d}y, \qquad a_{4,3} = \int_{\Omega_+} y\Delta Q(x,y)\mathrm{d}x\mathrm{d}y,$$

(these quantities are bounded by Corollary 5.3) $\mathbf{a}_7 = (a_{1,3}, 0, 0, a_{4,3}, 0, 0)$ and $\mathbf{a}_8 = (\frac{a_1^2}{4\sqrt{\pi x_0}}, 0, 0, a_1\mathcal{P}_0 a_3 \ln(x_0), 0, a_1\mathcal{P}_0 a_3)$. We then write $U_9(x) = \sum_{i=1}^{10} U_{9,i}(x)$, with (see (5.8) for the definitions of $\mathcal{J}_{1,1}$ and $\mathcal{J}_{1,2}$)

$$U_{9,1}(x) = \mathcal{P}_0\mathcal{J}_1[K_c, Q_{\mathbf{a}_2}](x) - \mathcal{I}P_{\mathbf{a}_1}(x) - u_{\mathbf{a}_8}(x)$$

$$U_{9,2}(x) = -\mathcal{P}_0\mathcal{J}_1[K_8 + K_c, Q_{\mathbf{a}_2}](x), \qquad U_{9,3}(x) = -\mathcal{J}_{1,2}[K_8, \Delta Q](x)$$

$$U_{9,4}(x) = -\mathcal{J}_{1,1}[K_8, \Delta Q - \mathcal{M}_0(\Delta Q)](x) + \mathcal{J}_{1,1}[\partial_y K_8, \mathcal{M}_1(\Delta Q)](x),$$

$$U_{9,5}(x) = K_8(x - x_0) \int_{x_0}^{\frac{x+x_0}{2}} \mathrm{d}\tilde{x}\; \mathcal{M}_0(\Delta Q(\tilde{x})) - \mathcal{J}_{1,1}[K_8, \mathcal{M}_0(\Delta Q)](x),$$

$$U_{9,6}(x) = \partial_y K_8(x - x_0) \int_{x_0}^{\frac{x+x_0}{2}} \mathrm{d}\tilde{x}\; \mathcal{M}_1(\Delta Q(\tilde{x})) - \mathcal{J}_{1,1}[\partial_y K_8, \mathcal{M}_1(\Delta Q)](x),$$

$$U_{9,7}(x) = K_8(x - x_0) \int_{\frac{x+x_0}{2}}^{\infty} \mathrm{d}\tilde{x}\; \mathcal{M}_0(\Delta Q(\tilde{x})),$$

$$U_{9,8}(x) = \partial_y K_8(x - x_0) \int_{\frac{x+x_0}{2}}^{\infty} \mathrm{d}\tilde{x}\; \mathcal{M}_1(\Delta Q(\tilde{x})),$$

$U_{9,9}(x) = -K_8(x - x_0)a_{1,3} - \partial_y K_8(x - x_0)a_{4,3} - u_{\mathbf{a}_7}(x - x_0)$, $U_{9,10}(x) = u_{\mathbf{a}_7}(x - x_0) - u_{\mathbf{a}_7}(x)$ and $U_{9,11}(x) = u_{\mathbf{a}_7}(x) + u_{\mathbf{a}_8}(x) - u_{\mathbf{a}_6}(x)$.

[6] Except for symmetric flows where $\int_{\mathbf{R}} yQ(x,y)\, \mathrm{d}y = 0$ by symmetry.

As in Proposition 5.7, we first compute the Fourier transform $\hat{T}$ of $U_{9,1}(x)$:

$$\hat{T}(x,x_0,k) = \frac{a_1^2 e^{-k^2(x-\frac{x_0}{2})}\big(1-e^{-\frac{k^2 x_0}{2}}\big)}{4\sqrt{2\pi x_0}},$$

and get $\|U_{9,1}(x)\|_{\infty,\frac{3}{2}} \leq \|\hat{T}(x,x_0,\cdot)\|_{1,\frac{3}{2}} \leq C$ for all $x \geq 2x_0$. We then note that

$$\begin{aligned}
\|U_{9,2}(x)\|_{\infty,\frac{3}{2}} &\leq C \sup_{x\geq x_0} \langle x\rangle^{\frac{3}{2}} \int_{x_0}^{x} \mathrm{d}\tilde{x} \int_{-\infty}^{\infty} \mathrm{d}k \; \mathcal{P}_0 e^{\Lambda_-(x-\tilde{x})-\frac{k^2\tilde{x}}{4}} \Big(k^2 + k^4(x-\tilde{x})\Big)\tilde{x}^{-\frac{3}{2}} \\
&\leq C \sup_{x\geq x_0} \langle x\rangle^{\frac{3}{2}} \int_{x_0}^{x} \mathrm{d}\tilde{x} \min\Big(\frac{\langle x-\tilde{x}\rangle^{\frac{5}{2}}}{(x-\tilde{x})^4},\; \frac{1}{\tilde{x}^{\frac{3}{2}}} + \frac{x-\tilde{x}}{\tilde{x}^{\frac{5}{2}}}\Big)\tilde{x}^{-\frac{3}{2}} \leq C.
\end{aligned}$$

Using Lemma 4.1, Corollary 5.3, as well as $x \geq 2x_0$, we get

$$\|U_{9,3}(x)\|_{\infty} \leq C\,x \sup_{\xi\geq\frac{x+x_0}{2}} \|\Delta Q(\xi)\|_\infty \leq C\langle x\rangle^{-\frac{3}{2}+(1+\varepsilon)\varphi},$$

$$\|U_{9,4}(x)\|_{\infty,\frac{9}{8}-(1+\varepsilon)\varphi} \leq C \sup_{x\geq x_0} \int_{x_0}^{\frac{x+x_0}{2}} \mathrm{d}\tilde{x}\; \|\rho_{\frac{5}{4}-2\varphi(1+\varepsilon)}\Delta Q(\tilde{x})\|_1 \leq C \int_{x_0}^{\infty} \frac{\mathrm{d}\tilde{x}}{\langle\tilde{x}\rangle^{1+\frac{\varepsilon\varphi}{2}}},$$

$$\|U_{9,5}(x)\|_{\infty,\frac{3}{2}-(2+\varepsilon)\varphi} \leq C\langle x\rangle^{(1+\varepsilon)\varphi} \int_{x_0}^{\frac{x+x_0}{2}} \mathrm{d}\tilde{x} \int_{\tilde{x}}^{\frac{x+x_0}{2}} \mathrm{d}z\; \langle z\rangle^{-2+(1+\varepsilon)\varphi} \leq C$$

$$\|U_{9,6}(x)\|_{\infty,\frac{3}{2}-(2+\varepsilon)\varphi} \leq C\langle x\rangle^{-\frac{1}{2}+(1+\varepsilon)\varphi} \int_{x_0}^{\frac{x+x_0}{2}} \mathrm{d}\tilde{x} \int_{\tilde{x}}^{\frac{x+x_0}{2}} \mathrm{d}z\; \langle z\rangle^{-\frac{3}{2}+(1+\varepsilon)\varphi} \leq C,$$

$$\|U_{9,7}(x)\|_{\infty,\frac{3}{2}-(1+\varepsilon)\varphi} \leq C\langle x\rangle^{1-(1+\varepsilon)\varphi} \int_{\frac{x+x_0}{2}}^{\infty} \mathrm{d}\tilde{x}\; \langle\tilde{x}\rangle^{-2+(1+\varepsilon)\varphi} \leq C$$

$$\|U_{9,8}(x)\|_{\infty,\frac{5}{4}-2(1+\frac{3\varepsilon}{2})\varphi} \leq C\langle x\rangle^{\frac{1}{4}-2(1+\frac{3\varepsilon}{2})\varphi} \int_{\frac{x+x_0}{2}}^{\infty} \mathrm{d}\tilde{x}\; \langle\tilde{x}\rangle^{-\frac{5}{4}+2(1+\frac{3\varepsilon}{2})\varphi} \leq C.$$

To establish the estimates on $U_{9,5}$ and $U_{9,6}$, we used integration by parts,

$$\|\partial_x K_8(x)\|_\infty \leq \|K_6(x)\|_\infty + \|K_7(x)\|_\infty \leq C\langle x\rangle^{-\frac{3}{2}}\langle x_0\rangle^{\varphi}$$

and

$$\|\partial_x\partial_y K_8(x)\|_\infty \leq \|\partial_y K_6(x)\|_\infty + \|\partial_y K_7(x)\|_\infty \leq C\langle x\rangle^{-2}\langle x_0\rangle^{\varphi}$$

if $x > 0$. Finally, using Lemma A.10 and A.11, we have for all $x \geq 2x_0$ that $\|U_{9,9}(x)\|_{\infty,\frac{3}{2}-\varphi} + \|U_{9,10}(x)\|_{\infty,\frac{3}{2}} \leq C$. Simple comparison with Proposition 4.2 shows that $U_{9,11}(x) = 0$ so that $\mathbf{a}_7 + \mathbf{a}_8 = \mathbf{a}_6$ as claimed. □

6. Checking the applicability to the usual exterior problem

In this section, we prove Proposition 1.7. We use the notation $r = (x^2+y^2)^{\frac{1}{2}}$. From [1, 4, 7], we get that any "Physically Reasonable" (PR) solution satisfies the estimates

$$|u(x,y)| \leq C \begin{cases} r^{-\frac{1}{2}} & \text{if } r \geq C \\ r^{-\min(\frac{1+\sigma}{2},1-\varepsilon)} & \text{if } 1-\cos(\phi) \geq r^{-1+\sigma} \end{cases} \tag{6.1}$$

$$|v(x,y)| \leq Cr^{-1}\ln(r), \quad |\partial_y u(x,y)| \leq Cr^{-1}\ln(r)^2, \quad |\partial_y v(x,y)| \leq Cr^{-\frac{3}{2}}\ln(r)^2$$

$$\omega(x,y) = c_1\partial_x\big(e^{\frac{x}{2}}\mathrm{K}_0(r)\big) + c_2\partial_y\big(e^{\frac{x}{2}}\mathrm{K}_0(r)\big) + \mathcal{O}\Big(e^{\frac{x-r}{4}}r^{-\frac{3}{2}}\ln(r)^2\Big),$$

$$\partial_y\omega(x,y) = c_1\partial_y\partial_x\big(e^{\frac{x}{2}}\mathrm{K}_0(r)\big) + c_2\partial_y^2\big(e^{\frac{x}{2}}\mathrm{K}_0(r)\big) + \mathcal{O}\Big(e^{\frac{x-r}{4}}r^{-2}\ln(r)^2\Big),$$

where ε is arbitrarily small, $\sigma \in [0,1]$, $\tan(\phi) = \frac{y}{x}$, c_1 and c_2 are constants and K_0 is Bessel's modified function of the second type of order zero. From this, we get immediately $|||(u,v,\omega)||| \leq C$ if x_0 is sufficiently large and $r > (2\min(\eta,\xi))^{-1}$ (using also $\ln(x) \leq C\langle x\rangle^{\varphi}$). Namely, for the estimates of the velocity fields u and v, the only difficulty is to prove that $\|u\|_{q,\frac{1}{2}-\frac{1}{q}} \leq C$. This follows easily upon splitting the integral in two regions where $|y| \leq cx$ and $|y| \geq cx$, and using that for $\sigma = \frac{1}{q}$, $\varepsilon = \frac{1}{2} - \frac{1}{2q}$ and x_0 sufficiently large, it follows from (6.1) that we have

$$|u(x,y)| \leq C\begin{cases} r^{-\frac{1}{2}} & \text{if } x \geq x_0 \text{ and } |y| < cx \\ r^{-\frac{1}{2}(1+\frac{1}{q})} & \text{if } x \geq x_0 \text{ and } |y| \geq cx \end{cases}.$$

For the estimates on the vorticity, we use $|z|^p e^{-z} \leq C_p$ for all $p \geq 0$ and the asymptotic development of K_0, so that for $x \geq x_0$ sufficiently large, we have

$$|\omega(x,y)| \leq Ce^{\frac{x}{4}-\frac{r}{4}}r^{-\frac{3}{2}}\big(|y| + \ln(x)^2\big), \qquad |\partial_y\omega(x,y)| \leq C\Big(e^{\frac{x}{4}-\frac{r}{4}}r^{-\frac{3}{2}}\Big).$$

This shows at once that $\|\partial_y\omega\|_{\infty,\frac{3}{2}} \leq C$. Then, for all $\alpha \geq 0$, after the change of variable $y = \sqrt{2xz+z^2}$ and using again that $|z|^p e^{-z} \leq C_p$, we get that

$$\|\partial_y\omega\|_{1,1} \leq C\sup_{x\geq x_0}\langle x\rangle\int_0^\infty \frac{\mathrm{d}z\ e^{-\frac{z}{4}}}{\sqrt{z}\sqrt{x+z}\sqrt{2x+z}} \leq C,$$

$$\|\rho_\alpha\omega\|_{2,\frac{3}{4}-\frac{\alpha}{2}} \leq C\sup_{x\geq x_0}\langle x\rangle^{\frac{3}{4}-\frac{\alpha}{2}}\Big(\int_0^\infty \mathrm{d}z\ \frac{e^{-\frac{z}{2}}}{\sqrt{z}}\frac{\big(\ln(x)^2+\sqrt{z}\sqrt{2x+z}\big)^2(z(2x+z))^\alpha}{(x+z)^2\sqrt{2x+z}}\Big)^{\frac{1}{2}},$$

as this last quantity is bounded for $\alpha = 0$ and $\alpha = \beta$, we get $|||(0,0,\omega)||| \leq C$. We then note that for $|y| \geq cx \geq cx_0$ with x_0, we have for all $q > 1$ $|u(x,y)|+|v(x,y)| \leq Cr^{-\frac{1}{2}(1+\frac{1}{q})}$ from which we deduce that $\|\rho_{\frac{1}{2}}u(x)\|_4 + \|\rho_{\frac{1}{2}}v(x)\|_4 \leq C$. Finally, it follows from e.g. [7], Section X.6, that there exist constants $\mathbf{m} = (m_1,m_2)$ such that for all $|y| \geq cx \geq cx_0$, we have for all $\varepsilon_1 > 0$ that

$$|u(x,y) - u_{\mathbf{m}}(x,y)| + |v(x,y) - v_{\mathbf{m}}(x,y)| \leq C|y|^{-\frac{3}{2}+\varepsilon_1}, \tag{6.2}$$

where $u_{\mathbf{m}}$ and $v_{\mathbf{m}}$ are defined in terms of Oseen's tensor $\mathbf{E}$ by

$$\begin{pmatrix} u_{\mathbf{m}}(x,y) \\ v_{\mathbf{m}}(x,y) \end{pmatrix} = \mathbf{m}\cdot\mathbf{E}(x,y). \tag{6.3}$$

From the explicit form of Oseen's tensor and (6.2), we get from (6.3) that

$$\|\rho_{\frac{1}{2}-(1+\varepsilon)\varphi}\mathcal{S}u(x)\|_1 + \|\rho_{\frac{1}{2}-(1+\varepsilon)\varphi}\mathcal{S}v(x)\|_1 \le C.$$

This completes the proof of Proposition 1.7.

7. Estimates on the boundary data

In this section we prove Theorem 1.8. As in Sections 4 and 5, the letter C stands for a constant which may change its value from place to place and does not depend on x_0.

Proof of Theorem 1.8. Let u, v and ω satisfy $|\!|\!|(u,v,\omega)|\!|\!| \le C$. The functions ν and w are determined by the evaluation of (2.1) at $x = x_0$, which gives

$$\begin{aligned} \mathcal{L}_u w + \nu &= u(x_0) - u_N(x_0), \qquad \mathcal{L}_v w + \mathcal{H}\nu = v(x_0) - v_N(x_0), \\ w &= \omega(x_0) - \omega_N(x_0). \end{aligned} \tag{7.1}$$

Note that as the stationary Navier–Stokes system is elliptic, the system (7.1) is overdetermined. Nevertheless, since we know that the solution exists, the three relations have to be satisfied. We now use this extra freedom to derive properties on ν and w. Define $U = u(x_0) - u_N(x_0)$ $V = v(x_0) - v_N(x_0)$ and $W = \omega(x_0) - \omega_N(x_0)$. By (3.6), since $\kappa > 0$, we have $\|(U,V,W)\| \le C$, so that

$$\|(\mathcal{L}_u w + \nu, \mathcal{L}_v w + \mathcal{H}\nu, w)\| \le C. \tag{7.2}$$

In particular, it implies at once that $\|(0,0,w)\| \le C$. Then, by interpolation, we have $\|\tilde{\mathcal{L}_u} w\|_{p,\frac{1}{2}-\frac{1}{2p}} \le \|\tilde{\mathcal{L}_u} w\|_1 + \|\tilde{\mathcal{L}_u} w\|_{\infty,\frac{1}{2}}$, where $\tilde{\mathcal{L}_u} = \mathcal{L}_u + \mathcal{I}\mathcal{P}_0$. Similarly, we have $\|\mathcal{L}_v w\|_{p,1-\frac{1}{2p}-\varphi} \le \|\mathcal{L}_v w\|_{1,\frac{1}{2}-\varphi} + \|\mathcal{L}_v w\|_{\infty,1-\varphi}$. Using these inequalities, $-\frac{1}{p} \le -\frac{1}{2p}$ and Lemma 3.5, we get $\|(\tilde{\mathcal{L}_u} w, \mathcal{L}_v w, 0)\| \le C_1\|(0,0,w)\| \le C$, and from (7.2), we get

$$\|(\nu - \mathcal{I}\mathcal{P}_0 w, \mathcal{H}\nu, w)\| \le C. \tag{7.3}$$

In particular, this implies that $\mathcal{H}\nu \in \mathrm{L}^p \cap \mathrm{L}^\infty$ and $\partial_y \mathcal{H}\nu \in \mathrm{L}^r$, which gives $\nu \in \mathrm{L}^p \cap \mathrm{L}^\infty$ (see Lemma 7.1 below). Since $q \ge p$, we get $\nu \in \mathrm{L}^q$, and then (7.3) also implies that $\mathcal{I}\mathcal{P}_0 w \in \mathrm{L}^q$ (because $\nu \in \mathrm{L}^q$ and $\nu - \mathcal{I}\mathcal{P}_0 w \in \mathrm{L}^q$). Thus $\mathcal{I}\mathcal{P}_0 w$ has to decay as $|y| \to \infty$, though maybe only in a weak sense. On the other hand, from the definition of $\mathcal{I}$ (see (1.14)), we have $\lim_{y\to\pm\infty} \mathcal{I}\mathcal{P}_0 w(y) = \pm\mathcal{M}_0(\mathcal{P}_0 w)$ (the limit exists since $(1+\rho_\beta)\omega \in \mathrm{L}^2$ implies $w \in \mathrm{L}^1$). This is compatible with $\mathcal{I}\mathcal{P}_0 w \in \mathrm{L}^q$ only if $\mathcal{M}_0(\mathcal{P}_0 w)$ vanishes. We can thus use Lemma 3.2 and get that

$$\|\mathcal{I}\mathcal{P}_0 w\|_1 \le C\big(\|w\|_{2,\frac{3}{4}}\big)^{1-\frac{3}{2\beta}}\big(\|\rho_\beta w\|_{2,\frac{3}{4}-\frac{\beta}{2}}\big)^{\frac{3}{2\beta}} \le C\|(0,0,w)\|.$$

Using again Lemma 3.5, we get $\|(\mathcal{L}_u w, \mathcal{L}_v w, 0)\| \le C_2\|(0,0,w)\|$, so that again from (7.2), we get $\|(\nu, \mathcal{H}\nu, w)\| \le C$. To complete the proof of Theorem 1.8, we still have to prove that (1.19) holds. This is done in Proposition 7.2 below. □

Lemma 7.1. *Let $p, q > 1$. There exists a constant $C_{p,q}$ such that for all f satisfying $(f, \partial_y f) \in \mathrm{L}^p \cap \mathrm{L}^\infty \times \mathrm{L}^q$, we have $(\mathcal{H}f, \partial_y \mathcal{H}f) \in \mathrm{L}^p \cap \mathrm{L}^\infty \times \mathrm{L}^q$ and $\|\mathcal{H}f\|_\infty \leq C_{p,q}(\|f\|_p + \|\partial_y f\|_q)$.*

Proof. Note that $\mathcal{H}f \in \mathrm{L}^p$ and $\partial_y \mathcal{H}f \in \mathrm{L}^q$ for $1 < p, q < \infty$ is a classical result which follows from Lemma 3.4 (see page 311). Then, if $q' \equiv \frac{q}{q-1} \geq p$, the L^∞ estimate follows from $\|\mathcal{H}f\|_\infty \leq (\|\mathcal{H}f\|_{q'} \|\partial_y \mathcal{H}f\|_q)^{\frac{1}{2}} \leq C(\|f\|_{q'} \|\partial_y f\|_q)^{\frac{1}{2}}$. However the $q' \geq p$ restriction is not essential: the Cauchy–Schwartz inequality and integration by parts gives

$$
\begin{aligned}
|\mathcal{H}f(y)| &= \left| \lim_{\varepsilon \to 0} \int_{|z| \geq \varepsilon} \frac{f(y-z)}{z} \mathrm{d}z \right| \leq C\|f\|_p + \left| \lim_{\varepsilon \to 0} \int_{\varepsilon \leq |z| \leq 1} \frac{f(y-z)}{z} \mathrm{d}z \right| \\
&\leq C\|f\|_p + \lim_{\varepsilon \to 0} \left| \ln(\varepsilon) \int_{y-\varepsilon}^{y+\varepsilon} \partial_z f(z) \mathrm{d}z \right| + \left| \int_{-1}^{1} \ln|z| \partial_y f(y-z) \mathrm{d}z \right| .
\end{aligned}
$$

The proof then follows from Hölder's inequality. □

Proposition 7.2. *Assume that $\|\rho_{\frac{1}{2}} \mathbf{v}(x_0)\|_4 + \|\rho_{\frac{1}{2}-(1+\varepsilon)\varphi} \mathcal{S}\mathbf{v}(x_0)\|_1 < \infty$, then for all $\varepsilon > 0$, it holds*

$$
\|\rho_{\frac{1}{2}-(1+\varepsilon)\varphi} \mathcal{S}\nu\|_1 + \|\rho_{\frac{1}{2}-(1+\varepsilon)\varphi} \mathcal{S}\mathcal{H}\nu\|_1 < \infty .
$$

Proof. In this proof, for concision, the letter C denotes a constant which depends on x_0, $\|\rho_{\frac{1}{2}} \mathbf{v}(x_0)\|_4$, $\|\rho_{\frac{1}{2}-(1+\varepsilon)\varphi} \mathcal{S}\mathbf{v}(x_0)\|_1$, $\|(\nu, \mathcal{H}\nu, w)\|$ and $\|\!|(u, v, \omega)|\!\|$. We will use that $\|\rho_a f\|_p \leq \|f\|_p^{1-a} \|\rho_1 f\|_p^a$ for all $p \geq 1$ and $0 \leq a \leq 1$, as well as $\|\rho_{\frac{1}{2}-(1+\varepsilon)\varphi} f\|_1 \leq C\|\rho_{1-(1+\frac{\varepsilon}{2})\varphi} f\|_2$ or $\|\rho_{\frac{1}{2}-(1+\varepsilon)\varphi} f\|_1 \leq C\|\rho_1 f\|_2$. We first note that by Lemma 3.5 and 3.2, (using also that the symbols $\tilde{\mathcal{L}}_u$ and $\mathcal{L}_v$, together with their derivatives w.r.t. the Fourier variable 'k' are bounded), we have $\|\rho_{\frac{1}{2}-(1+\varepsilon)\varphi} \mathcal{L}_v w\|_1 \leq \|\rho_1 \mathcal{L}_v w\|_2 \leq C$ and

$$
\|\rho_{\frac{1}{2}-(1+\varepsilon)\varphi} \mathcal{L}_u w\|_1 \leq \|\rho_{\frac{1}{2}-(1+\varepsilon)\varphi} \mathcal{I}\mathcal{P}_0 w\|_1 + \|\rho_1 \tilde{\mathcal{L}}_u w\|_2 \leq C .
$$

Then, using Lemma A.3 and $\|\rho_1 S\|_2 \leq \|\rho_{\frac{1}{2}} u(x_0)\|_4^2 + \|\rho_{\frac{1}{2}} v(x_0)\|_4^2 \leq C$, we get

$$
\begin{aligned}
&\|\rho_{\frac{1}{2}-\varepsilon} \mathcal{L}_1 S\|_1 \leq \|\rho_1 (\mathcal{L}_1 - \mathbb{1})\|_2 \|S\|_1 + \big(1 + \|(\mathcal{L}_1 - \mathbb{1})\|_1\big) \|\rho_1 S\|_2 \leq C , \\
&\|\rho_{\frac{1}{2}-\varepsilon} \mathcal{L}_2 S\|_1 \leq \|\rho_1 \mathcal{L}_2\|_2 \|S\|_1 + \|\mathcal{L}_2\|_1 \|\rho_1 S\|_2 \leq C .
\end{aligned}
$$

The same estimates hold for $\|\rho_{\frac{1}{2}-\varepsilon} \mathcal{L}_1 R\|_1$ and $\|\rho_{\frac{1}{2}-\varepsilon} \mathcal{L}_2 R\|_1$. We then use Proposition 3.12 and $v_{N,8}(x) = \omega_{N,4}$ to bound the contribution of $v_{N,8}$. Then, there are exponents $p \geq 0$ and $q < 1$ such that $\|K_2\|_{1,\{p,q\}}$, $\|K_3\|_{1,\{p,q\}}$, $\|K_4\|_{1,\{p,q\}}$, $\|K_5\|_{1,\{p,q\}}$, $\|\rho_1 K_2\|_{2,\{p,q\}}$, $\|\rho_1 K_3\|_{2,\{p,q\}}$, $\|\rho_1 K_4\|_{2,\{p,q\}}$ and $\|\rho_1 K_5\|_{2,\{p,q\}}$ are bounded. Using $\|\rho_{\frac{1}{2}-(1+\varepsilon)\varphi} f\|_1 \leq \|\rho_1 f\|_2$, this shows that the contributions of $v_{N,5}$ and $u_{N,5}$ is also bounded. For the contribution of $v_{N,4}$ and $u_{N,4}$, we note that

$$
\|\mathcal{S} K_{13}^*(\tilde{x} - x) Q(\tilde{x})\|_2 \leq C|\tilde{x} - x|^{-\frac{1}{2}} \langle \tilde{x} \rangle^{-\frac{3}{2}+\varphi} ,
$$

$$
\begin{aligned}
\|\rho_1\big(\mathcal{P}K_{13}^*(\tilde{x}-x)Q(\tilde{x})\big)\|_2 &\leq C\big(|\tilde{x}-x|^{\frac{1}{2}}\langle\tilde{x}\rangle^{-\frac{3}{2}+\varphi}+\langle\tilde{x}\rangle^{-\frac{5}{4}+\varphi}\big),\\
\|\rho_1\mathcal{S}\mathcal{P}_0K_{13}^*(\tilde{x}-x)Q(\tilde{x})\|_2 &\leq \Big(\int_{-\infty}^{\infty}\!\mathrm{d}k\,\Big(\partial_k e^{-|k||\tilde{x}-x|}\Big)^2|Q(\tilde{x},k)|^2\Big)^{\frac{1}{2}}\\
&\quad+\Big(\int_{-\infty}^{\infty}\!\mathrm{d}k\, e^{-|k||\tilde{x}-x|}|\partial_k\big(i\sigma(Q(\tilde{x},k)-Q(\tilde{x},-k))\big)|^2\Big)^{\frac{1}{2}}\\
&\leq C\big(|\tilde{x}-x|^{\frac{1}{2}}\langle\tilde{x}\rangle^{-\frac{3}{2}+\varphi}+\langle\tilde{x}\rangle^{-\frac{5}{4}+\varphi}\big),
\end{aligned}
$$

where we used that $|Q(\tilde{x},k)-Q(\tilde{x},-k)|\leq|k|^{\frac{1}{2}-\varepsilon}\|\rho_{\frac{1}{2}-\varepsilon}Q(\tilde{x})\|_1\leq|k|^{\frac{1}{2}-\varepsilon}\|\rho_1Q(\tilde{x})\|_2$, so that the coefficient of the Dirac measure appearing when differentiating σ w.r.t. k in the above expression vanishes. Since

$$
\|\rho_{\frac{1}{2}-(1+\varepsilon)\varphi}f\|_1\leq\|f\|_2^{(1+\frac{\varepsilon}{2})\varphi}\|\rho_1 f\|_2^{1-(1+\frac{\varepsilon}{2})\varphi},
$$

this implies finally that

$$
\begin{aligned}
\|\rho_{\frac{1}{2}-(1+\varepsilon)\varphi}\mathcal{S}K_{13}^*(\tilde{x}-x)Q(\tilde{x})\|_1 &\leq C|\tilde{x}-x|^{\frac{1}{2}-(1+\frac{\varepsilon}{2})\varphi}\langle\tilde{x}\rangle^{-\frac{3}{2}+\varphi}\\
&\quad+C|\tilde{x}-x|^{-\frac{1}{2}(1+\frac{\varepsilon}{2})\varphi}\langle\tilde{x}\rangle^{-\frac{5}{4}+\frac{6-\varepsilon}{8}\varphi}\,.
\end{aligned}
\tag{7.4}
$$

The same estimate holds for $\|\rho_{\frac{1}{2}-(1+\varepsilon)\varphi}\mathcal{S}K_{12}^*(\tilde{x}-x)Q(\tilde{x})\|_1$. Since $\varepsilon>0$, integrating (7.4) from $\tilde{x}=x_0$ to $\tilde{x}=\infty$ completes the proof. □

Acknowledgements. This work was completed during the stay of the author at the *Institut Fourier* in Grenoble and at the *Fields Institute* in Toronto under a grant from the Fond National Suisse pour la Recherche Scientifique, whose support is gratefully acknowledged. The author greatly benefited from the hospitality of these institutes and wishes in particular to express his gratitude to Jerry L. Bona, Walter Craig, Thierry Gallay, Jean-Claude Saut and Peter Wittwer for helpful discussions.

Appendix A. Kernels estimates

In this section we estimate the kernels in various L^p and Sobolev spaces of the 'y' variable. Here, the letter C stands for a constant which is independent of $\tau>0$. As the Kernels are most conveniently expressed in terms of their Fourier transform (though it is sometimes possible to calculate explicitly their inverse Fourier transform), we will estimate the norms in Fourier space as often as possible, using the following (classical) Lemma.

Lemma A.1. *Let $\beta>\frac{1}{2}$. There exists a constant C_β such that for all f with $\|(1+\rho_\beta)f\|_2<\infty$, we have $\|f\|_1\leq C_\beta\|f\|_2^{1-\frac{1}{2\beta}}\|\rho_\beta f\|_2^{\frac{1}{2\beta}}$, and in particular we have $\|f\|_1\leq C(\|f\|_2\|\rho_1 f|_2)^{\frac{1}{2}}\leq C(\|\hat{f}\|_2\|\hat{f}'\|_2)^{\frac{1}{2}}$ and $\|\partial_y f\|_1\leq C\|k\hat{f}\|_2^{\frac{1}{2}}(\|\hat{f}\|_2+\|k\partial_k\hat{f}\|_2)^{\frac{1}{2}}$, where $\hat{f}$ denote the (continuous) Fourier transform of f. Finally, for* *all* $s_1 \in [0,3]$ *and* $s_2 \in [0,2]$, *we have* $\|\rho_{s_1} f\|_2 \leq \|f\|_2^{1-\frac{s_1}{3}} \|\rho_3 f\|_2^{\frac{s_1}{3}}$ *and* $\|\rho_{1+s_2} f\|_2 \leq \|\rho_1 f\|_2^{1-\frac{s_2}{2}} \|\rho_3 f\|_2^{\frac{s_2}{2}}$.

Proof. Let $a > 0$, then

$$\|f\|_1 \leq \|(a+\rho_\beta) f\|_2 \|(a+\rho_\beta)^{-1}\|_2 \leq C_\beta \Big(a^{\frac{1}{2\beta}} \|f\|_2 + a^{\frac{1}{2\beta}-1} \|\rho_\beta f\|_2 \Big)$$

for some finite C_β. Setting $a = \|\rho_\beta f\|_2 / \|f\|_2$ completes the proof of the first inequality. The other ones follow from easy algebra, and Plancherel's and Young's inequalities. $\square$

We then introduce the functions

$$B_{\mu,\varphi}(x, n\tau) = \int_{-\infty}^{\infty} \mathrm{d}k \, \frac{|k|^{\varphi+2\mu} e^{2 \,\mathrm{Re}(\Lambda_-)x}}{|\Lambda_0|^{2\mu}} \,, \qquad B_\varphi(x, n\tau) = \int_{-\infty}^{\infty} \mathrm{d}k \, \frac{|k|^{2\varphi} e^{2 \,\mathrm{Re}(\Lambda_-)x}}{|\Lambda_0|^{2+2\varphi}}.$$

through which most estimates on the kernels can be easily obtained, and which satisfy the

Lemma A.2. *Let* $\mu \geq \frac{1}{2}$. *Then for all* $\varphi \geq 0$, *there exists a constant* C_φ *such that for all* $1 \leq \xi_1 \leq \mu + \frac{1}{2}$ *we have*

$$B_{0,\varphi}(x, n\tau) \leq C_\varphi \frac{e^{b(n\tau)x} \langle x \rangle^{\frac{\varphi+1}{2}}}{x^{\varphi+1}}, \qquad B_{\mu,\varphi}(x, n\tau) \leq C_\varphi \frac{e^{b(n\tau)x} \langle x \rangle^{\frac{\varphi}{2}}}{x^{\xi_1+\varphi}},$$

$$B_\varphi(x, n\tau) \leq C_\varphi \frac{e^{b(n\tau)x}}{\langle x \rangle^{\frac{1}{2}+\varphi}}$$

for all $x \geq 0$ *and* $n\tau \in \mathbf{R}$.

Proof. The first inequality follows from the estimate $\sup_{\zeta \geq 0} \frac{\zeta e^{b(n\tau)\zeta}}{(1+\zeta) c(n\tau)} \leq C$ applied to

$$B_{0,\varphi}(x, n\tau) \leq C e^{2b(n\tau)x} \left(\int_{|k|>1} \mathrm{d}k \, |k|^\varphi e^{-|k|x} + \int_{|k|\leq 1} \mathrm{d}k \, |k|^\varphi e^{-2c(n\tau)xk^2} \right).$$

Then, we note that since $\left|\frac{k}{\Lambda_0}\right|$ is uniformly bounded in k and $n\tau$, we trivially have $B_{\mu,\varphi}(x, n\tau) \leq C_\mu B_{0,\varphi}(x, n\tau)$ for all $\mu \geq 0$. To get the more precise bound of the Lemma in the case $\mu \geq \frac{1}{2}$, we use that $\left|\frac{k}{\Lambda_0}\right| \leq C$ and that by hypothesis on ξ_1, we have $0 \leq \xi_1 - 1 \leq 2\xi_1 - 1 \leq 2\mu$, hence

$$\begin{aligned} B_{\mu,\varphi}(x, n\tau) &\leq C e^{2b(n\tau)x} \left(\int_{|k|>1} \mathrm{d}k \, |k|^{\varphi+\xi_1-1} e^{-|k|x} + \int_{|k|\leq 1} \mathrm{d}k \, \frac{|k|^{\varphi+2\xi_1-1} e^{-2c(n\tau)xk^2}}{(1+(n\tau)^2)^{\frac{\mu}{2}}} \right) \\ &\leq \frac{C}{x^{\xi_1+\varphi}} \Big(e^{2b(n\tau)x} + \frac{(c(n\tau)^{-1}x)^{\frac{\varphi}{2}} e^{2b(n\tau)x}}{c(n\tau)^{\xi_1} (1+(n\tau)^2)^{\frac{\mu}{2}}} \Big). \end{aligned}$$

Since $c(n\tau)^{-\mu-\frac{1}{2}}(1+(n\tau)^2)^{-\frac{\mu}{2}} \leq C$ by hypothesis on μ and ξ_1, this completes the proof of the second inequality if $\varphi = 0$. We use again $\sup_{\zeta \geq 0} \frac{\zeta e^{b(n\tau)\zeta}}{(1+\zeta) c(n\tau)} \leq C$ to conclude in the case $\varphi > 0$. For the last inequality, we first note that $B_\varphi(x, n\tau) \le C_\varphi B_0(0, n\tau)$ (this follows again because $\left|\frac{k}{\Lambda_0}\right|$ is uniformly bounded). Then we have $B_0(0, n\tau) \le C$, so we only have to show that $B_\varphi(x, n\tau)$ decays at least like $e^{b(n\tau)x} x^{-\frac{1}{2}-\varphi}$ as $x \to \infty$, and this follows since

$$\begin{aligned} B_\varphi(x, n\tau) &\le C e^{2b(n\tau)x} \Big(\int_{|k|\le 1} \mathrm{d}k \, \frac{|k|^{2\varphi} e^{-2c(n\tau)xk^2}}{(1+(n\tau)^2)^{\frac{1+\varphi}{2}}} + \int_{|k|>1} \mathrm{d}k \, \frac{|k|^{2\varphi} e^{-|k|x}}{1+k^2} \Big) \\ &\le \frac{C e^{b(n\tau)x}}{x^{\frac{1}{2}+\varphi}} \left(\left(c(n\tau)^{\frac{1}{2}+\varphi} (1+(n\tau)^2)^{\frac{1+\varphi}{2}} \right)^{-1} + x^{-\frac{1}{2}-\varphi} \right). \qquad \square \end{aligned}$$

Note that in the bound on $B_{\mu,\varphi}(x)$ in Lemma A.2, the best decay rate as $x \to \infty$ improves as μ grows. The 'free' parameter ξ_1 gives a way to limit the growth of the divergence rate as $x \to 0$. We now turn to the estimates per se.

Lemma A.3. *Let* $\hat{\mathcal{L}}_1 = \frac{k^2}{k^2+(n\tau)^2}$ *and* $\hat{\mathcal{L}}_2 = \frac{|k| n\tau}{k^2+(n\tau)^2}$, *then* $|\rho_1(\mathcal{L}_1 - \mathbb{1})|_2 + |\rho_1 \mathcal{L}_2|_2 + |\mathcal{L}_1 - \mathbb{1}|_1 + |\mathcal{L}_2|_1 \le C$. *In particular,* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *are* $\mathrm{L}^p \to \mathrm{L}^p$ *bounded operators for all* $1 \le p \le \infty$.

Proof. The proof follows immediately using Fourier transforms: for fixed n, it holds $\|\hat{\mathcal{L}}_1 - 1\|_{\mathrm{L}^2} + \|\hat{\mathcal{L}}_2\|_{\mathrm{L}^2} \le C|n\tau|$ and $\|\partial_k(\hat{\mathcal{L}}_1 - 1)\|_{\mathrm{L}^2} + \|\partial_k \hat{\mathcal{L}}_2\|_{\mathrm{L}^2} \le C|n\tau|^{-1}$. $\square$

Lemma A.4. *For all* $p > 1$, $q \ge 2$ *and* $m \in \mathbf{N}$, *there exists a constant* $C > 0$ *such that* $\|\mathcal{P} K_{12}\|_{1,\{0,\frac{1}{4}\}} \le C|\tau|^{-\frac{1}{4}}$ *and* $\|\mathcal{P} K_{13}\|_{1,\{0,\frac{1}{4}\}} \le C|\tau|^{-\frac{1}{4}}$. *Furthermore, the following quantities are bounded* $\|\mathcal{P}_0 K_{12}\|_{p,\{0,1-\frac{1}{p}\}}$, $\|\mathcal{P}_0 K_{13}\|_{p,\{0,1-\frac{1}{p}\}}$, $\|\partial_y^m K_{12}\|_{q,\{0,1+m-\frac{1}{q}\}}$, $\|\partial_y^m K_{13}\|_{q,\{0,1+m-\frac{1}{q}\}}$, $\|\langle \tau x\rangle \mathcal{P} \partial_y^m K_{12}\|_{q,\{0,1+m-\frac{1}{q}\}}$ *and* $\|\langle \tau x\rangle \mathcal{P} \partial_y^m K_{13}\|_{q,\{0,1+m-\frac{1}{q}\}}$.

Proof. After the change of variables $k = \xi/x$, we get

$$\begin{aligned} \|\langle x\tau\rangle \partial_y^m K_{12}\|_{q,\{0,1+m-\frac{1}{q}\}} &\le \sup_{x\ge 0} \sup_{n\in\mathbf{Z}} \left(\int_{-\infty}^{\infty} \mathrm{d}\xi \, \left(\frac{\xi^{2+2m}(1+(\tau x)^2) e^{-2|\xi|}}{\xi^2 + (n\tau x)^2} \right)^{\frac{q}{2(q-1)}} \right)^{\frac{q-1}{q}} \\ &\le \left[\int_{|\xi|\le 1} \mathrm{d}\xi \, |\xi|^{\frac{qm}{q-1}} e^{-\frac{q|\xi|}{q-1}} + \int_{|\xi|\ge 1} \mathrm{d}\xi \, |\xi|^{\frac{q(1+m)}{q-1}} e^{-\frac{q|\xi|}{q-1}} \right]^{\frac{q-1}{q}}, \\ \|\partial_y^m K_{12}\|_{q,\{0,1+m-\frac{1}{q}\}} &\le \sup_{x\ge 0} \sup_{n\in\mathbf{Z}} \left(\int_{-\infty}^{\infty} \mathrm{d}\xi \, |\xi|^{\frac{qm}{q-1}} e^{-\frac{q|\xi|}{q-1}} \right)^{\frac{q-1}{q}}, \end{aligned}$$

for any $m \in \mathbf{N}$ and $q \ge 2$. The same holds for K_{13}. Since $K_{13} = -i\sigma K_{12}$, we have

$$\begin{aligned} \partial_k K_{13}(x,k) &= -i\delta(k) K_{12}(x,k) - i\sigma \partial_k K_{12}(x,k) \\ &= \frac{-i\delta(k)}{1 - \frac{in\tau}{|k|}} - i\sigma \partial_k K_{12}(x,k) = -i\delta_{n,0} - i\sigma \partial_k K_{12}(x,k), \end{aligned} \tag{A.1}$$

where $\delta_{n,0} = 1$ if $n = 0$ and $\delta_{n,0} = 0$ if $n \neq 0$. We thus have $\partial_k \mathcal{P}_0 K_{13}(x,k) \notin \mathrm{L}^2$, so that we cannot use Lemma A.1 to bound $\|\mathcal{P}_0 K_{13}(x)\|_{\mathrm{L}^1}$. In fact, $\mathcal{P}_0 K_{12}$ and $\mathcal{P}_0 K_{13}$ can be explicitly computed, giving $\mathcal{P}_0 K_{12}(x,y) = \frac{1}{\pi}\frac{x}{x^2+y^2}$ and $\mathcal{P}_0 K_{13}(x,y) = \frac{1}{\pi}\frac{y}{x^2+y^2}$. This shows that $\mathcal{P}_0 K_{13}(x,y) \notin \mathrm{L}^1$, and gives an easy way to prove the estimate on $\|\mathcal{P}_0 K_{12}\|_{p,\{0,1-\frac{1}{p}\}} + \|\mathcal{P}_0 K_{13}\|_{p,\{0,1-\frac{1}{p}\}}$ for $p > 1$ in direct space. On the other hand, (A.1) shows that $|\mathcal{P}\partial_k K_{13}(x)|_2 = |\mathcal{P}\partial_k K_{12}(x)|_2$, and we have

$$\begin{aligned}\|\partial_k \mathcal{P} K_{12}\|_{2,\{0,0\}} &= \sup_{x\geq 0}\sqrt{x}\sup_{n\in\mathbf{Z},n\neq 0}\left(\int_{-\infty}^{\infty}\mathrm{d}\xi\;\frac{e^{-2|\xi|}\big(\xi^4+(n\tau x)^2(1-|\xi|)^2\big)}{(\xi^2+(n\tau x)^2)^2}\right)^{1/2} \\ &\leq \sup_{x\geq 0}\frac{\sqrt{x}}{2}\left(\int_{|\xi|\leq 1}\mathrm{d}\xi\;\frac{1}{\xi^2+(\tau x)^2} + \frac{C}{\langle \tau x\rangle^2}\int_{|\xi|\geq 1}\mathrm{d}\xi\; e^{-|\xi|}\right)^{1/2} \leq \frac{C}{\sqrt{|\tau|}}.\end{aligned}$$

The estimates on $\|\mathcal{P}K_{12}\|_{1,\{0,\frac{1}{4}\}} + \|\mathcal{P}K_{13}\|_{1,\{0,\frac{1}{4}\}}$ are then proved using Lemma A.1. □

Lemma A.5. *The following quantities are bounded for all $1 \leq \beta \leq 3$:*

$$\begin{gathered}\|K_1\|_{1,\{0,0\}},\ \|K_1\|_{\infty,\{\frac{1}{2},1\}},\ \|\partial_y K_1\|_{1,\{\frac{1}{2},1\}},\ \|\partial_y K_1\|_{\infty,\{1,2\}},\ \|\partial_y^2 K_1\|_{\infty,\{\frac{3}{2},3\}}, \\ \|\partial_y^2 K_1\|_{1,\{1,2\}},\ \|\rho_\beta K_1\|_{2,\{-\frac{1}{4}+\frac{\beta}{2},0\}},\ \|\rho_\beta \partial_y K_1\|_{2,\{-\frac{3}{4}+\frac{\beta}{2},0\}},\ \|K_2\|_{1,\{0,\frac{1}{2}\}}, \\ \|K_2\|_{\infty,\{0,1\}},\ \|\rho_\beta K_2\|_{2,\{-\frac{3}{4}+\frac{\beta}{2},0\}},\ \|\partial_y K_2\|_{\infty,\{\frac{1}{2},2\}},\ \|\partial_y K_2\|_{1,\{\frac{1}{2},\frac{3}{2}\}}, \\ \|K_3\|_{1,\{0,\frac{1}{2}\}},\ \|\rho_1 K_3\|_{2,\{0,\frac{1}{4}\}},\ \|K_4\|_{1,\{0,\frac{1}{2}\}},\ \|\rho_1 K_4\|_{2,\{0,\frac{1}{4}\}},\ \|K_5\|_{1,\{\frac{1}{4},\frac{1}{4}\}}\end{gathered}$$

and $\|\rho_1 K_5\|_{2,\{\frac{1}{2},\frac{1}{4}\}}$. The same properties holds with K_n replaced by $e^{-\frac{b(\tau)x}{4}}\mathcal{P}K_n$ for $n = 1,2,3,4,5$.

Proof. We have $\left|\partial_k e^{\Lambda_- x}\right| \leq \frac{|k| x e^{\mathrm{Re}(\Lambda_-)x}}{|\Lambda_0|}$ and

$$\begin{aligned}\left|\partial_k^3 e^{\Lambda_- x}\right| &\leq e^{\mathrm{Re}(\Lambda_-)x}\Big(\tfrac{x^3|k|^3}{|\Lambda_0|^3} + \tfrac{x^2|k|}{|\Lambda_0|^2} + \tfrac{x|k|}{|\Lambda_0|^3}\Big) \\ \left|\partial_k^3 (k e^{\Lambda_- x})\right| &\leq e^{\mathrm{Re}(\Lambda_-)x}\Big(\tfrac{x^3|k|^4}{|\Lambda_0|^3} + \tfrac{x^2|k|^2}{|\Lambda_0|^2} + \tfrac{x}{|\Lambda_0|}\Big) \\ \left|\partial_k\big(\tfrac{k}{\Lambda_0}e^{\Lambda_- x}\big)\right| &\leq e^{\mathrm{Re}(\Lambda_-)x}\Big(\tfrac{c_1}{|\Lambda_0|} + \tfrac{c_2 k^2 x}{|\Lambda_0|^2}\Big), \\ \left|\partial_k^3\big(\tfrac{k}{\Lambda_0}e^{\Lambda_- x}\big)\right| &\leq e^{\mathrm{Re}(\Lambda_-)x}\Big(\tfrac{x^3|k|^4}{|\Lambda_0|^4} + \tfrac{x^2|k|^2}{|\Lambda_0|^3} + \tfrac{x}{|\Lambda_0|^2} + \tfrac{1}{|\Lambda_0|^3}\Big), \\ \left|\partial_k\big(\tfrac{k^2}{\Lambda_0}e^{\Lambda_- x}\big)\right| &\leq |k| e^{\mathrm{Re}(\Lambda_-)x}\Big(\tfrac{c_1}{|\Lambda_0|} + \tfrac{c_2 k^2 x}{|\Lambda_0|^2}\Big).\end{aligned}$$

Similarly, we have

$$\begin{aligned}|K_3(x,k)| + |K_4(x,k)| &\leq \frac{|k| e^{\mathrm{Re}(\Lambda_-)x}}{|\Lambda_0|}, \\ |\partial_k K_3(x,k)| + |\partial_k K_4(x,k)| &\leq e^{\mathrm{Re}(\Lambda_-)x}\Big(\tfrac{c_3}{|\Lambda_0|} + \tfrac{c_4 k^2 x}{|\Lambda_0|^2}\Big),\end{aligned}$$

$$|K_5(x,k)| \le Ce^{\mathrm{Re}(\Lambda_-)x},$$
$$|\partial_k K_5(x,k)| \le Ce^{\mathrm{Re}(\Lambda_-)x}\Big(\tfrac{1}{|\Lambda_0|} + \tfrac{|k|x}{|\Lambda_0|}\Big).$$

Finally, we note that for fixed x and n and $m = 1, 2$, we have

$$|\partial_y^m K_1|_1 \le C\left(|\partial_y^m K_1|_2^2 \left(m|\partial_y^{m-1}K_1|_2^2 + x^2|\partial_y^m K_2|_2^2\right)\right)^{\frac{1}{4}} .$$

The proof is completed using Lemma A.2 and $\mathcal{P}e^{\frac{b(n\tau)x}{4}} \le e^{\frac{b(\tau)x}{4}}$, we omit the details. $\square$

Lemma A.6. *For all $1 \le \beta \le 3$, $\frac{1}{4} \le \xi_2 \le 1$ and $1 \le \xi_3 \le \frac{5}{2}$, there exists a constant $C > 0$ such that $\|K_6\|_{1,\{0,\xi_2\}} + \|K_6\|_{2,\{0,\frac{\xi_3}{2}\}} + \|K_6\|_{\infty,\{\frac{1}{2},2\}} + \|\partial_y K_6\|_{\infty,\{1,3\}} + \|\rho_\beta K_6\|_{2,\{-\frac{5}{4}+\frac{\beta}{2},0\}} + \|\rho_\beta \partial_y K_6\|_{2,\{-\frac{3}{4}+\frac{\beta}{2},1\}} + \|\partial_y K_6\|_{1,\{\frac{1}{4},\frac{1+\xi_3}{2}\}} \le C$. The same estimate holds with K_6 replaced by $e^{-\frac{b(\tau)x}{4}}\mathcal{P}K_6$.*

Proof. For any $0 \le \sigma \le 1$, we have

$$|K_6(x,k)| \le C\left|\frac{\mathrm{Re}(\Lambda_-)}{\Lambda_0}\right|^{1-\sigma} \frac{e^{\mathrm{Re}(\Lambda_-)x/2}}{(x|\Lambda_0|)^\sigma} \le C\frac{e^{\mathrm{Re}(\Lambda_-)\frac{x}{2}}}{(x|\Lambda_0|)^\sigma},$$
$$|\partial_k K_6(x,k)| \le C\Big(\frac{|k|}{|\Lambda_0|^2} + \frac{x|k\mathrm{Re}(\Lambda_-)|}{|\Lambda_0|^2}\Big)e^{\mathrm{Re}(\Lambda_-)x} \le C\frac{|k|}{|\Lambda_0|^2}e^{\mathrm{Re}(\Lambda_-)\frac{x}{2}},$$
$$\left|\partial_k^3 K_6(x,k)\right| \le C\Big(\frac{x^2|k|^3}{|\Lambda_0|^4} + \frac{x|k|}{|\Lambda_0|^3} + \frac{|k|}{|\Lambda_0|^4}\Big)e^{\mathrm{Re}(\Lambda_-)\frac{x}{2}} .$$

Let $1 \le \xi_3 \le \frac{5}{2}$, $\sigma_3 = \frac{\xi_3}{2} - \frac{1}{4}$ and $\gamma_3 = \frac{\xi_3}{2} - \frac{1}{2}$. Since $0 \le \sigma_3, \gamma_3 \le 1$, for any fixed x, we have

$$|K_6(x)|_2^2 \le C\sup_{n\in\mathbf{Z}}\left(x^{-2\sigma_3}\int_{|k|\le 1}\mathrm{d}k\,\frac{e^{b(n\tau)x - c(n\tau)xk^2}}{(1+(n\tau)^2)^{\frac{\sigma_3}{2}}} + x^{-2\gamma_3}\int_{|k|>1}\mathrm{d}k\; e^{b(n\tau)x - \frac{|k|x}{2}}\right)$$
$$\le C\sup_{n\in\mathbf{Z}} e^{b(n\tau)x}\left(\frac{x^{-\frac{1}{2}-2\sigma_3}}{(1+(n\tau)^2)^{\frac{4\sigma_3-1}{8}}} + x^{-1-2\gamma_3}\right) \le Cx^{-\xi_3}.$$

The bound on $\|K_6\|_{1,\{0,\xi_2\}} + \|K_6\|_{2,\{0,\frac{\xi_3}{2}\}} + \|\rho_\beta K_6\|_{2,\{-\frac{5}{4}+\frac{\beta}{2},0\}}$ is completed using Lemma A.1, A.2 and $\mathcal{P}e^{\frac{b(n\tau)x}{4}} \le e^{\frac{b(\tau)x}{4}}$. To bound $\|\partial_y K_6\|_{1,\{\frac{1}{2},1+\xi_2\}}$, we note that for fixed x

$$|\partial_y K_6(x)|_1 \le C\sup_{n\in\mathbf{Z}}\left(|kK_6(x)|_2^2\left(|K_6(x)|_2^2 + |k\partial_k K_6(x)|_2^2\right)\right)^{\frac{1}{4}} \le C\left(\tfrac{1+x}{x^{2\xi_3+2}}\right)^{\frac{1}{4}} . \quad \square$$

Lemma A.7. *Let $\tilde{K}_7 = e^{-\frac{b(\tau)x}{4}}K_7$. The following quantities are bounded: $\|\tilde{K}_7\|_{\infty,\{0,1\}}$, $\|\tilde{K}_7\|_{2,\{0,\frac{3}{4}\}}$, $\|\tilde{K}_7\|_{1,\{\frac{1}{8},\frac{5}{8}\}}$, $\|\rho_\beta\tilde{K}_7\|_{2,\{\frac{3}{8}+\frac{\beta}{8},-\frac{9}{8}+\frac{3\beta}{8}\}}$, $\|\partial_y\tilde{K}_7\|_{\infty,\{\frac{1}{2},2\}}$, $\|\partial_y\tilde{K}_7\|_{1,\{\frac{5}{8},\frac{13}{8}\}}$ for all $1 \le \beta \le 3$.*

Proof. We have $|K_7(x,k)| \leq C\frac{|n\tau|}{\langle n\tau\rangle}e^{\mathrm{Re}(\Lambda_-)x}$, and

$$\begin{aligned}
|\partial_k K_7(x,k)| &\leq C\Big(\tfrac{|n\tau k|}{|\Lambda_0|^4}+\tfrac{x|kn\tau|}{\langle n\tau\rangle|\Lambda_0|}\Big)e^{\mathrm{Re}(\Lambda_-)x} \leq \tfrac{\langle x\rangle|n\tau|}{\langle n\tau\rangle}\tfrac{|k|e^{\mathrm{Re}(\Lambda_- x)}}{|\Lambda_0|},\\
\left|\partial_k^3 K_7(x,k)\right| &\leq C\tfrac{|n\tau|}{\langle n\tau\rangle}e^{\mathrm{Re}(\Lambda_-)x}\left(\tfrac{x^3|k|^3}{|\Lambda_0|^3}+\tfrac{x^2|k|}{|\Lambda_0|}+\tfrac{1+x}{|\Lambda_0|}\right).
\end{aligned}$$

We then note that $|\partial_y K_7|_1 \leq C|kK_7|_2^{\frac{1}{2}}\left(|K_7|_2+|k\partial_k K_7|_2\right)^{\frac{1}{2}}$. The proof is completed using Lemma A.2 that $\mathcal{P}K_7 = K_7$ and $\mathcal{P}e^{\frac{b(n\tau)x}{4}} \leq e^{\frac{b(\tau)x}{4}}$, and $\frac{|n\tau|\langle x\rangle^{\frac{1}{2}}}{\langle n\tau\rangle}e^{\frac{b(n\tau)x}{4}} \leq 2$. □

Lemma A.8. *Let $p \geq 2$ and $\tilde{K}_9 = e^{-\frac{b(\tau)x}{2}}K_9$. There exists a constant $C > 0$ such that $\|K_8\|_{\infty,\{\frac{1}{2},1\}} + \|K_8\|_{2,\{\frac{1}{4},\frac{1}{2}\}} + \|\partial_y K_8\|_{p,\{1-\frac{1}{2p},2-\frac{1}{p}\}} + \|\tilde{K}_9\|_{\infty,\{\frac{1}{2},1\}} + \|\tilde{K}_9\|_{2,\{\frac{1}{4},\frac{1}{2}\}} + \|\partial_y \tilde{K}_9\|_{p,\{1-\frac{1}{2p},2-\frac{1}{p}\}} \leq C$. Furthermore, for all $x \geq 0$, we have*

$$\|K_8(x,n\tau)\|_{\mathrm{L}^1} + \|\tilde{K}_9(x,n\tau)\|_{\mathrm{L}^1} \leq C\Big(1+\frac{\langle\tau\rangle}{|\tau|x^{\frac{1}{4}}}\Big).$$

The estimates of this lemma also hold with K_8 replaced by $e^{-\frac{b(\tau)x}{4}}\mathcal{P}K_8$.

Proof. We first note that $\mathcal{P}_0 K_9 = 0$ and $\mathcal{P}K_9 = K_9$. We then have $|K_8(x,k)| + |K_9(x,k)| \leq Ce^{\mathrm{Re}(\Lambda_-)x}$ and

$$|\partial_k K_8(x,k)| + |\partial_k K_9(x,k)| \leq Ce^{\mathrm{Re}(\Lambda_-)x}\Big(\frac{m_n}{|\Lambda_0|}+\frac{|k|x}{|\Lambda_0|}\Big),$$

with $m_n = 1$ if $n = 0$ and $m_n = \frac{\langle n\tau\rangle}{|n\tau|}$ if $n \neq 0$. We then get e.g.

$$\begin{aligned}
|\partial_k K_8(x)|_2^2 &\leq \sup_{n\in\mathbf{Z}}\left(\int_{|k|\leq 1}\mathrm{d}k\ \frac{m_n^2+k^2x^2}{\langle n\tau\rangle}e^{2\mathrm{Re}(\Lambda_-)x} + \int_{|k|>1}\mathrm{d}k\ \frac{m_n^2+k^2x^2}{1+k^2}e^{2\mathrm{Re}(\Lambda_-)x}\right)\\
&\leq C\sup_{n\in\mathbf{Z}}\left(e^{b(n\tau)x}\left(m_n^2+\sqrt{x}\right)\right).
\end{aligned}$$

The proof is completed using Lemmas A.1 and A.2, that $|n\tau|\langle n\tau\rangle^{-1}\langle x\rangle^{\frac{1}{2}}e^{\frac{b(n\tau)x}{4}} \leq 2$ and $\mathcal{P}e^{\frac{b(n\tau)x}{4}} \leq e^{\frac{b(\tau)x}{4}}$ (see also the proof of Lemma A.7), we omit the details. □

Lemma A.9. *Let $p \geq 2$ and $q \geq 1$. The following quantities are bounded:*

$$\begin{gathered}
\|e^{-\frac{b(\tau)x}{4}}K_{10}\|_{\infty,\{\frac{1}{2},1\}},\ \ \|e^{-\frac{b(\tau)x}{4}}K_{11}\|_{\infty,\{\frac{1}{2},1\}},\\
\|e^{-\frac{b(\tau)x}{4}}\partial_y K_{10}\|_{p,\{1-\frac{1}{2p},2-\frac{1}{p}\}},\ \ \|e^{-\frac{b(\tau)x}{4}}\partial_y K_{11}\|_{p,\{1-\frac{1}{2p},2-\frac{1}{p}\}}\,.
\end{gathered}$$

Furthermore, for all $x \geq 0$, we have

$$\begin{aligned}
|K_{10}(x)|_1 + |K_{11}(x)|_1 &\leq Ce^{\frac{b(\tau)x}{4}}\left(\frac{1}{x^{\frac{1}{2}}}+\frac{\langle x\rangle^{\frac{1}{8}}}{x^{\frac{1}{8}}}\left(1+\frac{1}{|\tau|\sqrt{x}}\right)^{\frac{1}{4}}\right),\\
|K_{10}(x)|_q + |K_{11}(x)|_q &\leq Ce^{\frac{b(\tau)x}{4}}\left(\frac{\langle x\rangle^{\frac{1}{2}-\frac{1}{2q}}}{x^{1-\frac{1}{2q}}}+\frac{\langle x\rangle^{\frac{1}{2}-\frac{3}{8q}}}{x^{1-\frac{7}{8q}}}+\frac{\langle x\rangle^{\frac{1}{2}-\frac{3}{8q}}}{|\tau|^{\frac{1}{4q}}x^{1-\frac{3}{4q}}}\right).
\end{aligned}$$

Proof. We first note that $\mathcal{P}_0 K_{11} = 0$, $\mathcal{P} K_{11} = K_{11}$. We then have $|K_{10}(x,k)| \leq C \frac{|n\tau|}{\langle n\tau\rangle} e^{\mathrm{Re}(\Lambda_-)x}$, $|K_{11}(x,k)| \leq C \frac{(n\tau)^2 e^{\mathrm{Re}(\Lambda_-)x}}{k^2+(n\tau)^2} \leq C \min\left(\frac{(n\tau)^2 e^{b(n\tau)x}}{k^2+(n\tau)^2},\ e^{\mathrm{Re}(\Lambda_-)x}\right)$ and

$$|\partial_k K_{10}(x,k)| \leq C e^{\mathrm{Re}(\Lambda_-)x} \tfrac{|n\tau|}{\langle n\tau\rangle} \left(\tfrac{1}{|\Lambda_0|} + \tfrac{|k|x}{|\Lambda_0|}\right),$$
$$|\partial_k K_{11}(x,k)| \leq C e^{\mathrm{Re}(\Lambda_-)x} \left(\tfrac{1}{|n\tau|} + \tfrac{|k|x}{|\Lambda_0|}\right).$$

This shows that

$$|K_{11}(x)|_\infty \leq C e^{\frac{b(\tau)x}{2}} \frac{\langle x\rangle^{\frac{1}{2}}}{x},$$
$$|K_{11}(x)|_1 \leq C \sup_{n\in\mathbf{Z}, n\neq 0} e^{\frac{b(n\tau)x}{2}} \min\left(|n\tau|, \tfrac{\langle x\rangle^{\frac{1}{2}}}{x}\right)^{\frac{1}{4}} \left(\tfrac{\langle x\rangle^{\frac{1}{2}}}{(n\tau)^2 x} + \sqrt{x}\right)^{\frac{1}{4}}.$$

The proof is completed using $|n\tau|^{-1} \leq C|\tau|^{-1}$ if $|n| \geq 1$, $|n\tau|\langle n\tau\rangle^{-1}\langle x\rangle^{\frac{1}{2}} e^{\frac{b(n\tau)x}{4}} \leq 2$ and $\mathcal{P} e^{\frac{b(n\tau)x}{4}} \leq e^{\frac{b(\tau)x}{4}}$ (see also the proof of Lemma A.7), we omit the details. □

Lemma A.10. *Let* $K_c(x,y) = \mathcal{P}_0 \frac{e^{-\frac{y^2}{4x}}}{\sqrt{4\pi x}}$. *We have*

$$\|\partial_y^m (K_1 - K_c)\|_{\infty,\{\frac{m+5}{2}, m+4\}} + \|\partial_y (K_1 - K_c)\|_{1,\{3,\frac{9}{2}\}} \leq C\tau^{-2}\langle\tau\rangle^2$$
$$\|\partial_y^m (K_8 - K_c)\|_{\infty,\{\frac{m+5}{2}, m+4\}} + \|K_2 - \partial_y K_c\|_{\infty,\{3,5\}} \leq C\tau^{-2}\langle\tau\rangle^2$$

for all $m \in \mathbf{N}$.

Proof. Let $\Delta K(x) = \mathcal{P}_0 K_1(x) - K_c(x)$. We first note that $\mathcal{P}_0|\Lambda_- + k^2| \leq Ck^4$, so that

$$\begin{aligned}
|\partial_y^m \Delta K(x)|_\infty &\leq \sup_{n\in\mathbf{Z}} \int_{-\infty}^{\infty} \mathrm{d}k\ |k|^m \mathcal{P}_0 e^{\mathrm{Re}(\Lambda_-)x} \left|1 - e^{-(k^2+\Lambda_-)x}\right| \\
&\leq C x B_{0,4+m}(x/2,0) \leq C\langle x\rangle^{\frac{m+5}{2}} x^{-m-4}, \\
|\partial_y^m \mathcal{P} K_1(x)|_\infty &\leq \sup_{n\in\mathbf{Z}} \int_{-\infty}^{\infty} \mathrm{d}k\ |k|^m \mathcal{P} e^{\mathrm{Re}(\Lambda_-)x} \leq C B_{0,m}(x/2,\tau) \\
&\leq C\langle x\rangle^{\frac{m+5}{2}} x^{-m-4} \sup_{x\geq 0}\left(x^3\langle x\rangle^{-2} e^{\frac{b(\tau)x}{4}}\right) \leq C\tau^{-2} \frac{\langle x\rangle^{\frac{m+5}{2}}}{x^{m+4}}, \\
|\partial_y^m \Delta K(x)|_2^2 &\leq \sup_{n\in\mathbf{Z}} \int_{-\infty}^{\infty} \mathrm{d}k\ |k|^{2m} \mathcal{P}_0 e^{2\mathrm{Re}(\Lambda_-)x} \left|1 - e^{-(k^2+\Lambda_-)x}\right|^2 \\
&\leq C x^2 B_{0,8+2m}(x,0) \leq C \frac{\langle x\rangle^{\frac{9+2m}{2}}}{x^{7+2m}}, \\
|\partial_y^m \mathcal{P} K_1(x)|_2^2 &\leq \sup_{n\in\mathbf{Z}} \int_{-\infty}^{\infty} \mathrm{d}k\ |k|^{2m} \mathcal{P} e^{2\mathrm{Re}(\Lambda_-)x} \leq C B_{0,2m}(x,\tau) \\
&\leq C\langle x\rangle^{\frac{9+2m}{2}} x^{-7-2m} \sup_{x\geq 0}\left(x^6\langle x\rangle^{-4} e^{\frac{b(\tau)x}{4}}\right) \leq C\tau^{-4} \frac{\langle x\rangle^{\frac{9+2m}{2}}}{x^{7+2m}},
\end{aligned}$$

$$\begin{aligned}
|\partial_y\big(y\Delta K(x)\big)|_2^2 &\leq Cx^2 \sup_{n\in\mathbf{Z}} \int_{-\infty}^{\infty}\mathrm{d}k\ |k|^4\mathcal{P}_0 e^{2\mathrm{Re}(\Lambda_-)x}\left|\tfrac{2\Lambda_-}{\Lambda_0}+1-e^{-(k^2+\Lambda_-)x}\right|^2 \\
&\leq C\big(x^2 B_{0,8}(x,0)+x^4 B_{0,12}(x,0)\big) \leq C\langle x\rangle^{\frac{13}{2}}x^{-9}, \\
|\partial_y\big(y\mathcal{P}K_1(x)\big)|_2^2 &\leq x^2 \sup_{n\in\mathbf{Z}} \int_{-\infty}^{\infty}\mathrm{d}k\ |k|^4\mathcal{P}e^{2\mathrm{Re}(\Lambda_-)x} \leq Cx^2 B_{0,4}(x,\tau) \\
&\leq C\langle x\rangle^{\frac{13}{2}}x^{-9}\sup_{x\geq 0}\big(x^6\langle x\rangle^{-4}e^{\frac{b(\tau)x}{4}}\big) \leq C\tau^{-4}\langle x\rangle^{\frac{13}{2}}x^{-9}\,.
\end{aligned}$$

The proof is completed using $\|\partial_y f\|_{\mathrm{L}^1} \leq \big(\|\partial_y f\|_{\mathrm{L}^2}(\|f\|_{\mathrm{L}^2}+\|\partial_y(yf)\|_{\mathrm{L}^2})\big)^{\frac{1}{2}}$, $K_2(x) = \partial_y(K_1(x)+K_6(x)+K_7(x))$ and $K_8(x) = K_1(x)+K_6(x)+K_7(x)$. □

Lemma A.11. *Let $K_c(x) = \frac{e^{-\frac{y^2}{4x}}}{\sqrt{4\pi x}}$, $K_0(x) = \frac{1}{\pi}\frac{x}{x^2+y^2}$, $\Delta K_c(x) = K_c(x-x_0) - K_c(x)$ and $\Delta K_0(x) = K_0(x-x_0) - K_0(x)$, then for all $m \in \mathbf{N}$ and $1 \leq s \leq \infty$, there exist constants C_m such that*

$$\|\partial_y^m \Delta K_c(x)\|_{s,\frac{3}{2}-\frac{1}{2s}+\frac{m}{2}} + \|\rho_1\partial_y^m \Delta K_c(x)\|_{2,\frac{3}{4}+\frac{m}{2}} \leq C_m\langle x_0\rangle, \tag{A.2}$$

$$\|\partial_y^m \Delta K_0(x)\|_{\infty,m+2} + \|\partial_y^m \mathcal{H}\Delta K_0(x)\|_{\infty,m+2} \leq C_m\langle x_0\rangle, \tag{A.3}$$

for all $x \geq 2x_0 \geq 2$.

Proof. Since $x - x_0 \geq \frac{x}{2}$ for $x \geq 2x_0 \geq 2$, we have

$$\|\partial_y^m \Delta K_c(x)\|_\infty \leq \int_{-\infty}^{\infty}\mathrm{d}k|k|^m\big|e^{-k^2(x-x_0)} - e^{-k^2x}\big| \leq x_0\int_{-\infty}^{\infty}\mathrm{d}k|k|^{m+2}e^{-\frac{k^2x}{2}}\,.$$

We proceed similarly for $\|\partial_y^m \Delta K_c(x)\|_2^2$ and $\|\rho_1\partial_y^m \Delta K_c(x)\|_2^2$. The proof of (A.2) is completed with the use of Lemma A.1. That of (A.3) is similar. □

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(accepted: June 1, 2005; published Online First: May 16, 2006)